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STRESS CONCENTRATION

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The present collection presents the papers given during a symposium on the concentration of stresses around holes in plates and shells. The results /2* derived from investigating the concentration of stresses near holes in plates and shells in linear and nonlinear arrangements are given, taking into consideration the effect of the material anisotropy, the relative arrangement of the holes, geometric and physical nonlinearity, plastic deformation, etc.

The article is intended for scientists and engineers interested in calculating the strength of thin-walled constructions of laminar and shell types weakened by holes.

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*Note: Numbers in the margin indicate pagination in the original foreign text.

PREFACE

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Strength research is coordinated in our country by 12 divisions. One of them is the division"Concentration of Stresses". The leadership of this division was entrusted to the Institute of Mechanics of the Academy of Sciences of the USSR.

Plans were made to hold yearly symposia devoted to separate branches of the problem. The first symposium convened on May 26-29, 1964, in Kiev. The following principal goals were formulated: (1) determination of the principal directions to be followed by research on the problem;(2) analysis and approval of a "List of Problems" in research on the concentration of stresses.

The proceedings of the symposium were conducted in plenary sessions and two sections: section of plates and shells, and section of plane problems. The delegates listened to 46 reports devoted to an investigation of the concentration of stresses near various holes, both free and reinforced. In the reports, the following important problems of stress concentration were covered: formulation of the basic equations of the problem of stress concentration around holes in plates and shells under simplifying assumptions, differing from the Kirchoff-Love hypothesis; concentrations of stresses around holes in a nonlinear arrangement for an elastic plastic problem of deformations; concentrations of stresses around holes in glass plastics; dynamic problems of a concentration of stresses; an investigation of the influence of cracks, macro- and micro-impurities on the concentration of stresses; experimental methods of investigating stress concentration.

During the symposium particular emphasis was given to the following requirements: 1) development of methods for investigating stress concentration in structural elements and parts of machines made of new synthetic materials, which change their physico-mechanical characteristics in the course of time; 2) a considerable intensification of experimental work designed to determine the physico-mechanical characteristics of these materials; 3) the development of effective methods for solving the problems of stress concentration, and a more rapid incorporation of the research results into engineering practice.

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Design of Shells of Revolution with a Small Hole at the Apex Under Symmetrical and Antisymmetrical Load. V.I. Kruglyakova, p. 145-151.

Propagation of an Elastic Expansion Wave From a Circular Hole in a Cylindrically Anisotropic, Inhomogeneous Plate. V.D. Kubenko, p. 152-159. 8 refs.

One Condition Sufficient for the Existence of a Secondary Load Producing Zero Moment State in a Shell. P.I. Kudrik, p. 160-164.

Generalization of the Griffith-Sneddon Criterion to the Case of an Inhomogeneous Elastic Body. V.I. Mossakovskiy, and M.T. Rybka, p. 165-172. 6 refs.

Stress State of a Plane Weakened by a Broken-Line Crack. V.I. Mossakovskiy, P.A. Zagubizhenko, and P.E. Berkovich, p. 173-177.

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Study of the Stress Concentration Near Holes in Plates During Bending. B.L. Pelekh, p. 184-189.

Effect of Foreign Macroinclusions on the Distribution of Temperature Fields and Stresses in Elastic Bodies. Ya.S. Podstrigach, p. 190-200. 27 refs.

Effect of a Diffusion Process on the Stress Concentration Near a Circular Hole. Ya. S. Podstrigach and V.S. Pavlina, p. 201-205.

Propagation of Elastic Waves Along a Cylindrical Cavity Filled with a Conducting Fluid. I.T. Selezov, p. 206-212. 11 refs.

A Case of Dynamic Stresses in an Unbounded Elastic Space with a Cylindrical Cavity. M.M. Sidlyar, p. 213-217. 5 refs.

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Bending of Reinforced Plates. V.I. Tul'chiy, p. 233-238. 9 refs.

A Method of Determining the Stress Concentration at "Nodal Points". A.G. Ugodchikov, p. 239-244.

Solution of Several Problems for Doubly Connected Regions with Contiguous Circular Boundaries. Yu.A. Ustinov, p. 245-249. 6 refs.

Numerical Method for Conformal Mapping of Simply and Multiply Connected Regions, Based on Trigonometric Interpolation. P.F. Fil'chakov, p. 250-260. 6 refs.

Complete Elimination of Stress Concentrations Around Holes in Plates. N.P. Fleyshman and B.L. Pelekh, p. 261-265. 9 refs.

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3 THE CONCENTRATION OF STRESSES AROUND CURVILINEAR HOLES IN PLATES AND SHELLS*

6 G. N. Savin (Kiev)

N67-24502

Plates and shells are the basic elements of modern construction which, for various reasons -- most often in order to decrease the weight of the construction -- are weakened by various holes. Therefore, a study of the distribution of stresses near holes is a very important problem, both from the theoretical and engineering point of view. /5

For the first time, a solution of the plane problem of the theory of elasticity regarding the distribution of stresses near a circular hole was published in 1898 by G. Kirsch (Ref. 163). The solution of this problem for a plate weakened by an elliptic hole -- where the hole was extended along the major axis of the ellipse -- was given for the first time in 1909 by G.V. Kolosov**

* A summary of this survey was presented by the author during the meeting of the Elasticity Theory Subsection of the Solid State Mechanics Section at the Second All-Union Conference on Theoretical and Applied Mechanics, convening in Moscow from January 29 till February 5, 1964.

** It can be seen from the words of Academician S. A. Chaplygin (Ref. 135) that the basic relations of the plane problem of the mathematical theory of elasticity were known to S. A. Chaplygin as early as in 1900. They have not been published, however, and therefore were not known. Consequently, they could not possibly influence the further development of the plane problem of the theory of elasticity.

(Refs. 61, 62). This work is remarkable in that it opened up a new era in the development of the plane problem of the mathematical theory of elasticity. This was due to the fact that G.V. Kolosov proposed formulating the plane problem of the theory of elasticity by using the methods of functions of a complex variable. He derived the basic relations for the plane problem of the theory of elasticity in terms of complex variables, thereby predeterming the further development of the problem for several decades in advance. In the further development of the /6
plane problem of the mathematical theory of elasticity, a basic contribution was made by Academician N.I. Muskhelishvili (Ref. 77) and his students [for summaries of their works, see (Refs. 20 and 142)]. Not wishing to repeat ourselves, we shall only point out that the work of the Soviet scientists, D. I. Sherman, S. G. Mikhlin, S. G. Lekhnitskiy (Ref. 67), G. N. Savin (Ref. 92), A. S. Kosmodamianskiy (Ref. 63), et al, was the most important factor in the development of methods for solving the plane problem in both isotropic and anisotropic media -- namely, in the creation of both exact and approximate methods for its solution.

Concentration of stresses around a curvilinear hole. In the investigation of stresses around any curvilinear hole, an especially effective method is that of N. I. Muskhelishvili (Ref. 77), based on the application of conformal mapping of the exterior of a given hole onto the exterior (or interior) of a unit circle.

As is known, by this method the solution of the problem amounts to finding two analytical functions $\phi(\zeta)$ and $\psi(\zeta)$ from the functional equations

$$\begin{aligned} \varphi(\zeta) + \frac{1}{2\pi i}\int_{\gamma} \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi'(\sigma)}\,\frac{d\sigma}{\sigma-\zeta} &= A(\zeta); \\ \psi(\zeta) + \frac{1}{2\pi i}\int_{\gamma} \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\varphi'(\sigma)\,\frac{d\sigma}{\sigma-\zeta} &= B(\zeta), \end{aligned} \tag{1}$$

where $\omega(\zeta)$ is the function specifying the conformal mapping of the exterior of unit circle γ onto the exterior of the hole under study bounded by outline Γ (Figure 1)

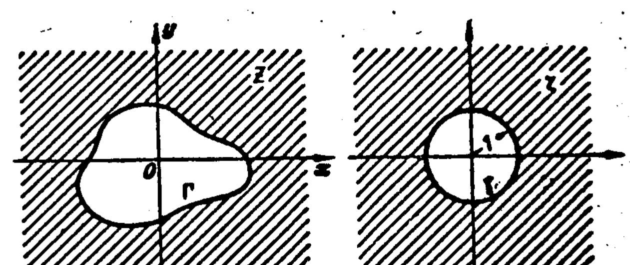


Figure 1

After the appearance in 1933 of the first edition of N.I. Muskhelishvili's famous monograph (Ref. 77) a vast number of problems on stress concentration -- besides different kinds of holes -- were solved by this method. In this process, a number of fundamental data were obtained on the nature of stress distribution around holes and, in particular, the effect on stress concentration exerted by such factors as the rounding of hole outlines, positioning of the hole with respect to stress field caused by an imposed load, anisotropy of the material, plastic zones, rigidity of reinforcing rings, etc. was demonstrated.

As a result, there has in the last thirty years been an accumulation of numerous works on this question, which include the solution of a great number of problems which are important from the viewpoint both of theory and, in particular, of practice. The list of these papers at present contains more than five hundred items. Therefore, without giving a complete list of the works, we will restrict ourselves merely to indicating the monographs (Refs. 15, 67, 92, 141) and surveys (Ref. 20, 118, 142, 154) which examine the results of the abovementioned papers or point out the sources where they are published. It should be noted that similar works are also appearing at present and will appear in the future, since a developing technology is continuously posing /7
ever new problems (Refs. 3, 16).

As is easily seen from functional equations (1), the solution of the stated problem essentially amounts to finding the function $\omega(\zeta)$. It is therefore not surprising that this method has engendered a huge amount of literature on the formulation of the function $\omega(\zeta)$ which conformally maps the exterior (or interior) of the unit circle onto the exterior of the hole under study, as well as literature on the development of methods -- at first purely analytical methods (Ref. 24), graphical numerical methods (Ref. 9), but later experimental and analytical methods -- using electromodeling of conformal mapping (Ref. 112). Recently the function $\omega(\zeta)$ has been determined by means of computers (Ref. 45). All of these matters are set forth in more detail in P. F. Fil'chakov's monograph (Ref. 117), which gives an extensive bibliography.

Therefore we have at our disposal powerful means of formulating conformal mapping functions for an extensive class of problems for holes whose boundaries have no corner points. In many cases, these methods give the function $\omega(\zeta)$ which effects an accurate (or very nearly accurate) mapping of the investigated region onto a unit circle -- e.g., increasing the number of terms in the mapping function successfully indicates the effect of hole outline curvature at any of its characteristic points -- usually in a corner point with a small radius of curvature -- on the coefficient of stress concentration. S. M. Belonosov (Ref. 8), however, using the integral transformations of Fourier and Mellin in conjunction with integrals of the Cauchy type, demonstrated that in some cases for regions with corner cusp points on their boundaries an increase in the number of terms in the polynomial mapping function may not always give the same picture of the stress state at the corner point near the hole as follows from a linear statement of the problem. Many plane problems in elasticity theory for doubly-connected regions have been solved by A. G. Ugodchikov (Ref. 112) and V. I. Makhovikov et al (Ref. 16) by mapping an annulus onto the given region. /8

Effect of material anisotropy. The effect of material anisotropy on stress concentration near an elliptical (in a particular case, a round) hole was first studied by S. G. Lekhnitskiy (Ref. 67). Later G. N. Savin (Ref. 92) examined this and other problems by a different method. Up till now, no accurate solution has been given for the other hole shapes. Applying the "perturbation method' (Ref. 75) to holes nearly elliptical or round in shape, S. G.

Lekhnitskiy (Ref. 67) has given an approximate solution to this problem for an anisotropic plate weakened by a square, triangular with rounded corners, or oval hole under various conditions "at infinity" (tension, pure flexure, etc.).

A. S. Kosmodamianskiy (Ref. 63) extended these solutions to encompass holes having other shapes -- rectangular, trapezoidal, arched, and isosceles triangular shapes. The problems which he solved made it possible for him to discover laws reflecting the effect of plate material anisotropy on the stress state occurring in the plate under uniaxial tension, in the case both of a free hole and of a hole which is reinforced by a rigid ring or which is filled with an elastic core.

Multiple-connected regions. Periodic problems. Studies of the stress state in plates and shells, which are both isotropic and anisotropic, weakened by several equal or unequal holes and, on the whole, are weakened by a finite or infinite series of holes, are of great interest in engineering.

The elastic equilibrium of shells having positive Gaussian curvature which are weakened by a series of curvilinear holes has been investigated by I. N. Vekua (Ref. 20).

A general solution of the plane problem of elasticity theory, both for an isotropic and for an anisotropic medium for any multiply-connected region has been given by D. I. Sherman (Ref. 142).

G. N. Savin (Ref. 92) has provided a solution for the periodic problem of elasticity theory in an isotropic medium for an infinite series of congruent holes which are equally loaded. G. N. Bukharin (Ref. 92) was the first one to inquire into the problem of stress distribution in a plate which is weakened by a large number of round holes. The plate which is weakened by a square grid containing round holes has been investigated by Ya. Dvorzhak (Ref. 41).

We encounter an additional development in the formulation of approximate solutions to these problems in the studies of A. S. Kosmodamianskiy (Ref. 63). In these studies, he investigates multiply-connected and periodic problems for holes having congruent shape, both for isotropic and anisotropic material of a plate which is weakened by one or more series of equal and equally-loaded holes.

Aiming at deriving efficient approximate solutions for specific hole shapes, /9
A. S. Kosmodamianskiy (Ref. 63) introduces certain simplifications into the general solutions of D. I. Sherman (Ref. 142). This makes it possible for him to propose a number of efficient approximate solutions for a plate weakened by a finite number of curvilinear holes. He has, in particular, studied problems of stress distribution beside two square holes with rounded corners; beside two unequal holes, one elliptical and the other square with rounded corners; and beside three round holes under uniaxial, as well as biaxial plate tension. Cases are examined of tension in an anisotropic plate with two or three round

holes; one, two, or three infinite series of equal elliptical holes; and a plate weakened by two unequal holes, one of which is elliptical and the other round.

A study of the solutions mentioned has made it possible to draw a number of interesting conclusions of a general nature: the picture of the stress state in a plate weakened by two, three, or an infinite number of round holes and under tension in one or two directions has enabled us to ascertain the effect exerted on stress concentration by the number of holes, their mutual disposition, plate material anisotropy, etc. The final formulas thus derived are sufficiently simple and therefore easy to use. The accuracy of these approximate solutions was estimated from the precision with which the boundary conditions found fulfill the conditions of the problem under study.

Dynamic problems of stress concentration and of the propagation of elastic waves from holes. Kromm's paper (Ref. 169) is devoted to the propagation of elastic perturbations in an infinite plate under uniform pressure suddenly applied to the edge of a round hole, or for the sudden generation of radial velocity at its boundary points. It was subsequently found that the corresponding problem of tangential perturbation is quite similar to the one described, as in the static case (Goodier and Johnsman [Ref. 158]).

These investigations indicate that the propagation of perturbations is of a wave nature, and that the displacements on the wave front are discontinuous, while the stresses and displacement velocity have a discontinuity acquiring values proportional to $1/\sqrt{r}$. As time passes, the stress state in the plate behind the wave front asympotically approaches the static state in which stresses are proportional to $1/r^2$. Miklowitz (Ref. 174) has dealt with the problem of a suddenly rupturing plate under tension in all directions. He showed that the relief wave propagating in this case gives an 11.5% increase /10 in stress concentration on the edge of the punctured hole, in comparison to the static case. The annular stresses which are generated may be conducive to formation and development of radial cracks.

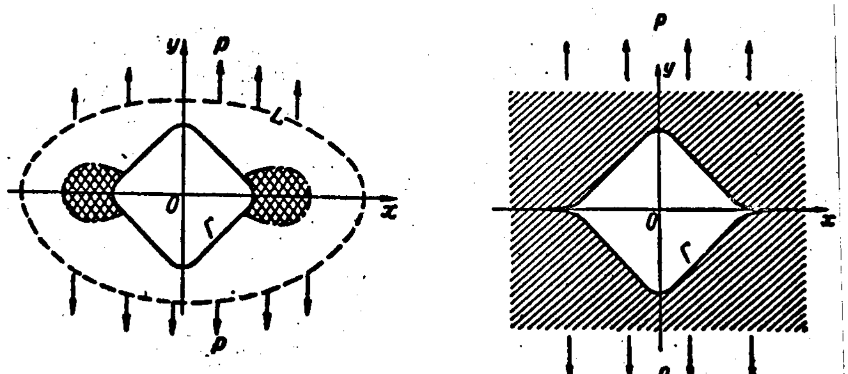
M. M. Sidlyar (Ref. 105, 106) examined the problem of stress concentration near a round hole in a plate under the influence of longitudinal forces transient in time applied to its edges. The problem of stress propagation on the edge of a hole as the result of a decreasing potential, harmonic elastic expansion wave has been examined by Pao Yi-Hsin (Ref. 82). He demonstrated that, for certain wavelengths and Poisson coefficient values, a rise in stress concentration beside the hole is detected. This solution, however, evokes some doubt, because the stress concentration coefficient obtained as the limiting case in the static problem proves to be dependent on the elastic constants of the medium.

R. D. Mindlin (Ref. 72) points out that the theory of generalized plane stress gives a sufficiently good definition of the wave process in a plate only for waves whose wavelength is considerably greater than the plate thickness.

Plane deformation, which is mathematically quite similar to the generalized plane stress state, is the subject of several more studies. We may mention papers by Baron and Mattews (Ref. 146), Baron (Ref. 6), and Baron and Parnes (Ref. 7), in which the cavity is defined as a right circular cylinder of a plane shock wave (wave front parallel to cavity axis). In the case of a decreasing expansion wave, the coefficient of stress concentration proves to be greater than the static one, $k=3.28$ (instead of 3) when $\theta = \pi/2$.

Durelli and Riley (Refs. 151-153) employed photoelastic methods to investigate stress distribution on the edge of round and elliptical holes when stress waves of long and short duration pass through the plate. They reached the conclusion that dynamic imposition of a load causes no great change in the magnitude (only 10-11%) of the stress or its distribution from the static case.

Elastic-plastic problems. Holes with cracks. In the increased stress region, either plastic zones may appear near the holes, which at first partially encompass the hole edge (Figure 2a, cross-hatching) and may completely encompass it only when the external load reaches the proper values, or cracks may appear which arise from embrittlement of the material (Ref. 113) at points of increased stress on the edge of the hole (Figure 2b), proceeding into the plate. The corresponding stress components located at the ends of these cracks, and found from theoretical solutions of the plane problem of elasticity theory, /11
become infinite. This shows that when such cracks are present the plate weakened by a hole in which they have appeared must be destroyed under any external force.



a b

Figure 2

Numerous experiments, however, indicate that by no means all cracks are dangerous, i.e., they result in immediate destruction of one portion. This is also corroborated by the theoretical studies of G. I. Barenblat and of his numerous successors (Refs. 4, 5, 113), M. Ya. Leonov (Ref. 66), V. V. Panasyuk (Ref. 80), and their collaborators. This particular research area of determining stress concentration coefficients around circular cracks is, in the last analysis, aimed at establishing the bearing capacity of solid bodies weakened by cracking.

G. I. Barenblat (Refs. 4 and 5) has formulated a theory which makes it possible to define the ultimate load in brittle fracture of a body with rather well developed microscopic cracks. The author of the works mentioned demonstrated that the ultimate equilibrium state of a cracked plate appears when the coefficient of stress concentration located in proximity to the point of the crack, and found by methods of classical elasticity theory, reaches a certain limiting value. Therefore the determination of stress concentration coefficients near the end region (extreme end) of the crack takes on particular interest.

(Ref. 66, 80) propose a new model of an ideally brittle body which makes it feasible to study the stress state in brittle fracture of a body with acicular stress concentrators, and to find the limiting equilibrium of this body in the case of arbitrary initial large and small cracks. Taking advantage of this /12
model, M. Ya. Leonov (Ref. 66), V. V. Panasyuk (Ref. 80), and P. M. Vitvitskiy (Ref. 23) gave a generalized solution of Griffiths' problem and Sack's problem.

O. L. Bowie (Ref. 147) and H. F. Bueckner (Ref. 149) have also studied brittle fracture in the case where radial cracks go out from the surface of a round hole in a plate.

(Ref. 147) deals with the problem of brittle fracture of an infinite body in the case where n equal radial cracks go out on the free surface of a circular cavity, while constant tensile forces are applied at infinity. Stress distribution in the examined region is found by N. I. Muskhelishvili's method. The critical loads necessary for crack development to start are determined by means of Griffiths' energy method. The computations are carried through to the end only for one or two cracks. It has been shown that the local stress field around the hole has an insignificant effect on development of sufficiently large cracks even when $\ell/r_h > 1$, where ℓ is crack length and r_h is hole radius. For small cracks, the critical load ratio for cases of multifold and uniaxial tension tends toward 2/3 -- i.e., toward the ratio of stress concentration coefficients for these cases of loading. This indicates the appreciable effect of the local stress field in development of small cracks.

(Ref. 23) handles the problem of tension exerted on an infinite plate with a round hole and two equal cracks by constant forces "at infinity" directed normal to the crack surface. In the solution, it was presupposed that fracture proceeds in accord with M. Ya. Leonov's simplified model of a brittle body (Ref. 66). The critical load was found from the conditions given in (Ref. 71).

It is theoretically demonstrated in the work by Ya. D. Fridman and Ye. M. Morozov (Ref. 12) that a plate with a round hole under uniform pressure applied to the edge of the hole is fractured with the formation of radial cracks.

If plastic deformations occur at several points on the hole, they will gradually develop into the region and along the hole edge as the external load grows larger. Boundary L (Fig. 2a) separating the plastic from the elastic zone is not known in advance; it must be determined by solving the problem.

This class of so-called elastic-plastic problems of stress concentration around holes is of great interest both in theory and in practice. We will not dwell on an analysis of the problems comprising the monograph (Ref. 92), noting only that these problems are discussed in papers by L. A. Galin, O.S. Parasyuk, G. N. Savin, and A. P. Sokolov. Let us also point out that the paper by L. A. Galin is the first in this direction for the case of plane deformation, and that of A. P. Sokolov the first for the case of a thin plate. /13

Let us now pause on the basic research in this class of problem which has been made in recent decades.

We find investigations of elastic-plastic problems for plates with a round hole completely enveloped in a plastic zone in works by G. Yu. Dzhanelidze (Ref. 42), K. N. Shevchenko (Ref. 140), B. V. Zaslavskiy (Ref. 44), I. Yu. Khoma (Ref. 123), B. Budyanskiy (Ref. 148), and O. L. Mangasarian (Ref. 172).

Non-circular, for the most part elliptical, hole problems are treated in papers by V. M. Panferov (Ref. 81), D. D. Ivlev (Refs. 46, 47), V. S. Sazhin (Ref. 104), and P. I. Perlin (Refs. 84, 85).

Let us briefly indicate the methods employed in the above-mentioned works. Panerov (Ref. 81) uses A. A. Il'yushin's method of plastic solutions; Ivlev (Refs. 46-47) employs the method of the small parameter characterizing deviation of the shape of the hole under investigation from the round shape. Perlin (Refs. 84-85) and Sazhin (Ref. 104) suggest an inverse formulation of the problem. This formulation assumes that the position of any two points on the boundary separating the elastic from the plastic zone is known and requires that the whole boundary between these zones be defined, as well as the stress "at infinity" causing the given elastic-plastic deformation beside the hole in question.

In connection with these problems, we must mention the studies of G. P. Cherepanov (Refs. 136-138), which expand upon the class of problems stipulating that the region in which their solution is sought is not known in advance and must be ascertained while they are being solved. Such problems include:

(1) The previously mentioned elastic-plastic problem (G. P. Cherepanov has proposed a new approach to solving this problem for the case of a round hole completely surrounded by a plastic zone);

(2) Local plate buckling around holes under tension because of loss of plate stability in these zones;

(3) The problem in elasticity theory which seeks the boundary of a body (or a part of this boundary) from conditions imposed on the stress distribution in this body.

A reverse problem in elasticity theory, as applied to the problem of stress concentration around holes, leads to a pragmatically very interesting problem involving calculation of the boundary of a hole in a plate with a given principal stress state in which stress concentration near this hole will be minimum or nonexistent.

This hole, as G. P. Cherepanov has shown in (Ref. 138), proves to be an ellipse in a plate under tension in a single direction. /14

The case of partial envelopment of the hole by the plastic zone is treated by P. I. Perlin (Refs. 84, 85) and I. I. Fayerberg (Ref. 115). Works by A. S. Kosmodamianskiy (Ref. 64) and I. Yu. Khoma (Ref. 124) deal with elastic-plastic problems with an infinite series of identical round holes.

Ring reinforcement of holes. Optimum rings. The first studies on reinforcement of holes in thin plates are those of S. P. Timoshenko (see Ref. 92), in whose work methods of material resistance based on the curved beam theory were used to inquire into problems of reinforcing round holes and square holes with rounded corners.

The methods of the elasticity theory were first applied by V. L. Fedorov [see (Ref. 92)] to the problem of subjecting an elastic plane with a circular hole to tension when an elastic ring has been soldered to this hole.

A great deal of attention has been paid during the last fifteen years to investigating the effect of reinforcing rings on stress distribution near holes. This was favored at the beginning of this period by development of powerful and efficient methods of solving the plane problem in elasticity theory (Ref. 77) making it possible not only to formulate this contact problem in the most general form, but also in many other cases -- i.e., for many particular types of holes -- to obtain an effective solution to it.

Owing to the resemblance between mathematical formulations of the plane problem in elasticity theory and the problem of thin plate bending, the above-indicated methods were also successfully transferred by A. I. Lur'ye the (Ref. 68) and S. G. Lekhnitskiy (Ref. 67) to bending problems for both isotropic and anisotropic plates.

The basis for extensive studies of stress concentration beside holes reinforced with elastic rings by application of complex-variable function methods was provided by S. G. Mikhlin's work (Ref. 73) which examines the elastic equilibrium of an inhomogeneous ring consisting of a series of concentric rings.

Using complex-variable function methods, G. N. Savin (Ref. 92), D. V. Vaynberg (Ref. 15), M. P. Sheremet'yev (Ref. 141), and other scientists have investigated an extensive class of contact problems of the elastic equilibrium of thin plates and slabs weakened by holes of circular and other shapes, with edges reinforced by elastic rings. They have also discussed contact boundary problems in the plane theory of elasticity and the theory of thin-plate bending for regions of different rigidity consisting of concentric zones. This made it possible to obtain solutions for a broad class of problems of importance for engineering.

The three cited works by Savin, Vaynberg, and Sheremet'yev (Ref. 15, 92, 141) give an extensive bibliography on this problem. A rather detailed survey of works by Soviet and foreign authors on this problem is also contained in J. G. Goodier's article (Ref. 28). There is therefore no need to repeat these /15 reviews, but it is more advisable to pause briefly on general statements of contact problems involving reinforcement of holes in thin plates and slabs, i.e., the possible variant schematizations of these problems in their mathematical formulation.

The early studies assumed that the elastic ring was rather wide and that its stress state was described by equations of the plane theory of elasticity, or else by the theory of plane plate bending. As thus stated, it was quite simple to solve problems for a simple or compound circular ring.

The problem of reinforcing non-circular holes is considerably more complex and may be simplified to render a more or less acceptable, but still approximate, solution possible by substantial restrictions on the shape of the ring. Such a simplification [e. g., (Ref. 141)] is possible for rings whose external and internal boundaries are represented by two coordinate lines derived from mappings of the function depending on the shape of the hole in question.

For thin reinforced rings or rings having a shaped/transverse cross-section, the reinforcement ring chosen was a thin curvilinear elastic rod of constant or varying cross-section and having an elastic behavior described by the theory of small deformations of thin curvilinear rods. This simplification of the contact problem made it possible for N. P. Fleyshman (Ref. 92) to study the effect of a round reinforcing ring for many particular cases and to find its parameters which are optimum (in a certain sense). This statement of the problem for reinforcing holes in the shape of ellipses, squares with rounded corners, etc. proved to be rather complex, and could only be solved [see (Ref. 14)] by the method of successive approximations.

Relaxation of the boundary conditions led to further simplification of the contact problem of reinforcing a plate or thin plate with a sufficiently thin curvilinear rib. Thus, for the plate it is assumed that the reinforcing rib reacts only to tension and compression in the case of the plane problem, while in thin plate bending it reacts to the deflection from its surface. With such a computational system, G. N. Savin and N. P. Fleyshman (Ref. 93) obtained a solution in quadratures to the combined contact problem for the exterior of an elliptical hole with a reinforced rim -- a ring of constant transverse cross-section. This solution permits generalization to the case of any smooth hole, /16
i.e., holes whose outlines contain no corner points.

There are thus three alternate versions of stating contact problems involving reinforcement of hole edges by elastic rings. The precise determination of the limits of applicability of each of the three varieties of computational systems is the object of the subsequent investigations. This determination may obviously be realized only on the basis of adequate experimental data and accurate solutions of a number of elasticity theory problems involving the joint effect of either a plate or a shell with a hollow cylinder sealed into its hole.

The ring for which there is no concentration of stresses around the hole in the plate or shell is commonly regarded as the optimum ring for reinforcing such a hole. This stress concentration will obviously always be absent when the rigidity of the reinforcing ring exactly equals that of the flat disk cut from the plate in question and having the shape of the hole outline. It is, however, not always possible to select this optimum ring under other conditions which restrict either its shape or the material from which it is made. In these cases, we may virtually speak of an approximate solution in which further assumptions -- for example, in regard to the stressed state in the ring -- may be made to simplify the problem.

In determining the optimum parameters of a ring reinforcing a circular hole in an extensible plate Mansfield (Ref. 173) assumed that the stressed state in the ring is momentless, since the ring works only during extension and compression.

Identifying the outline of the seal with the axis of the reinforcing ring, which he treats as an elastic thread which is only subjected to extension and bending, M. P. Sheremet'yev (Ref. 141) examined the problem of reinforcing a round hole in an isotropic plate under both unixial tension and under pure flexure. V. I. Tul'chiy (Ref. 111) regarded the reinforcing ring as a curved bar of variable rigidity, here assuming that ring thickness equals plate thickness, and demonstrated that in this case the width of the optimum reinforcing ring should satisfy an ordinary Abelian differential equation of the second kind. Hence, it follows that in this statement of the problem the optimum ring cannot always be realized.

If, however, the stresses in the ring due to bending moments are neglected -- i.e., if it is considered as without moment -- then, as Tul'chiy demonstrated (Ref. 111), it is always possible to determine the rigidity of the optimum ring both for an isotropic and an anisotropic plate. With these simplifications we may go one step farther, i.e., we may assume that ring rigidity under tension varies by the same law as does the stress component having the largest absolute value around an unreinforced hole of the shape under consideration. /17

If, then, we assume for a plate with a round hole subjected to uniaxial tension by forces $\sigma_x = P$ (Fig. 3) that the reinforcing ring rigidity EF for tension changes according to the law $EF = (EF)_1 + (EF)_2 \cos 2\theta$, with a steel ring having $b_{\theta=\frac{\pi}{2}} = 1.24$ cm and $b_{\theta=0} = 0.114$ cm, the stresses in a copper plate are reduced 20% over the case of a cross-section of the same weight. It is easy to ascertain that, with a given coefficient of stress concentration in a plate, reinforcing rings constructed in the manner mentioned above, which we will call quasi-optimum rings, will be lighter than rings of constant cross-section.

Since in practice any stresses (if the proper material is chosen) may be assumed in the ring, the choice of quasi-optimum ring may also vary in many ways. Of all modes, the best will be that in which the ring has the least weight.

Further development of the theory of optimum hole reinforcement is

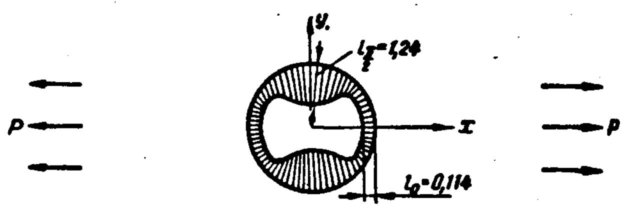


Figure 3

advisably directed toward solution of the following problems.

1. For a plate with a given hole and under the effect of a given system of external forces, let us select the optimum or quasi-optimum reinforcing ring which has the least weight.

Let us find the arrangement of a given number of holes of given shape, so that the optimum or quasi-optimum system of reinforcing rings will have the least weight.

Non-linear problems of stress concentration around holes in plates. In many areas of elasticity, non-linear problems are at present becoming urgent, both in statics and in dynamics. Among these problems are non-linear ones dealing with stress concentration along various holes. Therefore, serious efforts are being made to formulate and to solve problems of this sort under various assumptions with regard to type of non-linear stress-deformation relationship and geometric non-linearity of the problem. /18

No unified approach to these problems in the general case, however, has yet been worked out. The situation is a little better, in our opinion, with regard to the plane non-linear problem in elasticity theory. In the study of stress concentration near holes, several directions to be taken by research in such problems in elasticity theory have been proposed [see survey, (Ref. 96)].

1. General, geometrically plane and physically non-linear problem.

2. Physically non-linear, but geometrically linear plane problem.

3. Geometrically non-linear, but physically linear plane problem.

In each of the above directions, an approach has already been devised as well as mathematical methods of solving the pertinent problems.

Thus, the first direction includes the joint works of A. E. Green, J. E. Adkins, G. G. Nicholas, and R. T. Shield (Refs. 143-145, 159). Under the most general assumptions as to physical and geometrical non-linearity, they derived, using complex variables, a system of equations for plane non-linear theory, both for plane deformation and for the generalized stress state as well as incompressible and compressible materials.

An approximate method has been proposed -- the method of the small parameter, which for terms of the first order leads to the basic relationships of plane linear elasticity theory in complex form -- i.e., to the familiar Kolosov-Muskhelishvili ratios. For the successive approximations, it leads to the classic problems of linear elastic theory with their right sides depending on the terms of the preceding approximations. A system has been found for determining terms of the second order both for plane deformation and for a thin plate. An approximate solution is first given (accurate to terms of the second order) of the problem of stress concentration near a round hole, either free or with a rigid core sealed in, with the plate in the uniaxial stressed state "at infinity."

Subsequently, the paper by Yu. I. Koyfman and the author (Ref. 94) based on correlations of plane non-linear theory (Refs. 144, 145, 169) used the theory of complex variable functions to study certain correlations between terms of the second order and to formulate a statement of the different versions of the basic boundary value problems of the plane theory when the boundary of the region in the deformed or undeformed state is given. This paper has shown that finding the complex second order potentials determining the second-order stress functions in different versions of the first and second boundary /19 value problems may be reduced to solving boundary value problems in the theory of complex variable functions with the boundary conditions

$$m\varphi^{(2)}(t) + t\overline{\varphi^{(2)\prime}(t)} + \overline{\psi^{(2)}} - F(t, \bar{t}, \gamma, \delta) = f^{2}(t) \text{ on } \Gamma \qquad (2)$$

where $f^{(2)}(t)$ is the given function of external influences; $F(t, \bar{t}, \gamma, \delta)$ is the known function of first order terms, where the type of this function and the parameters γ, δ entering into it depend on the type of the basic boundary problem and its version; Γ is the boundary of the region in the deformed or undeformed state, depending on the type of problem; and $m = 1$ for the first basic problem, $m = \kappa$ for the second basic problem. In the same article (Ref. 94) an approximate solution, accurate to terms of the second order, is derived for the problem of stress concentration near a round hole in a thin plate whose edge is reinforced by a wide ring which is sealed in.

Yu. I. Koyfman's papers (Refs. 56-60) continued the study of the second approximation (terms of the second order) (Ref. 94) for certain problems on stress concentration around free and reinforced circular and elliptical holes in a sheet in a state of uniform stress at infinity. He investigated the stressed state near a round hole reinforced with a thin, linearly elastic ring and near round and elliptical holes into which absolutely rigid cores are sealed.

(Ref. 56-60 and 94) have demonstrated that in the general case of the general plane non-linear problem in elasticity theory the coefficient of stress concentration, found with an accuracy to terms of the second order, depends on the following factors: (a) initial and final hole shape, (b) degrees and type of external forces (tension or compression) at infinity, (c) elastic properties of the plate materials and (if the opening is reinforced) of the ring material, and (d) type of elastic equilibrium (plane deformation, generalized stress state).

The survey in (Ref. 96) gives a detailed analysis of the results of these and other papers with tables of the concentration coefficient values for round and elliptical holes. We shall therefore not pause to analyze these works, but shall merely point out that L. A. Tolokonikov (Refs. 109, 110) and V. G. Gromov and L. A. Tolokonnikov (Ref. 25) also give versions of the approach to solving problems of stress concentration near a round hole (plane deformation) when an incompressible medium is greatly deformed. There is also a somewhat different solution to the same displacement problem in the work of I. N. Slezinger and S. D. Barskaya (Ref. 107).

The physically non-linear (but geometrically linear) plane problem (plane deformation or generalized plane stress state) may be derived as a partial case from the general one indicated above, but there is no known work on the subject. This direction is at present encompassed by the papers of G. Kauderer (Ref. 52), who has given an equation for the solution of a certain type of non-linear elasticity ratios which are characteristic of many metals, in which the tension-compression curve deviates from a straight line even under comparatively small stresses (Fig. 4), and are characteristic of many materials (nonferrous metals, certain plastics, etc.) in which this curve is already seen to deviate perceptibly from Hooke's straight line. The small parameter method was used to solve this equation. /20

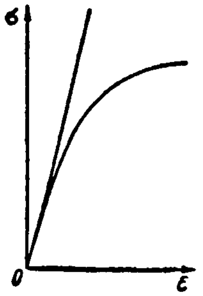


Figure 4

Based on this theory Yindr (Ref. 52) and I. A. Tsurpal (Refs. 127-134) made an approximate study (accurate to the second and third approximations) of stress concentration near a circular hole in a plate with several particular types of plate loading at infinity (tension, compression, pure shear, extension in all directions). I. A. Tsurpal has, in his papers (Refs. 132-134) formulated boundary conditions and coupling conditions, and solved several physically non-linear problems on the reinforcement of a circular hole in a plate with concentric, linearly elastic rings of different materials for a given force at infinity. From these studies it follows that calculation of the physical non-linearity of the material (in the indicated approximation) leads to a decrease -- in comparison to the classic case -- in stress concentration near the hole. In the case of a physically non-linear plate material, the stresses near the opening are more smoothly distributed than in the case of a physically linear material.

The author (Ref. 95), by using conformal mapping of the region outside of the hole in question to the exterior of a unit circle and by introducing Kolosov-Muskhelishvili complex potentials, gave a solution for the problem of stress concentration around curvilinear holes in a physically non-linear plate with non-linear elasticity ratios (Ref. 47). For the desired stress function represented as an expansion with respect to the small parameter, the differential equations and boundary conditions were obtained for each of the successive approximations in curvilinear coordinates given by the mapping funciton. Because of the cumbersome nature of the right sides of the differential equations obtained, however, this method results in very complex computations for a non-circular hole. Therefore, the work of A. N. Guz', G. N. Savin, and I. A. Tsurpal (Ref. 29) with the same non-linear law of elasticity (Ref. 47) proposes a new approach to solving the problem stated. /21

The theoretical basis of this approach may be found in the work of F. Stopelli (Ref. 177), who proved the theorem that a singular solution exists to the general equations of the three-dimensional non-linear theory of elasticity. He substantiated the feasibility of expanding displacement components in absolutely converging series with respect to the small parameter ε with a non-zero convergence radius, under the condition that there is a sufficiently smooth solution of the analogous problem in linear elasticity theory. This smooth solution is basic in the method of solution proposed by Guz' et al. (Ref. 29).

The essence of the method is that for the holes obtained from the mapping function,[1]

$$z^* = R_0[\zeta + \varepsilon f(\zeta)], \tag{3}$$

when $\rho = 1$, where $\varepsilon < 1$ is a small parameter; R_0, a constant describing the size of the hole and its position relative to the coordinate axes; and

$$f(\zeta) = \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \cdots + \frac{a_n}{\zeta^n};$$

$$z^* = r^* e^{i\varphi}; \quad r^* = R_0 r; \quad z = re^{i\varphi}; \quad \zeta = \rho e^{i\theta}; \quad a_i = \text{const}.$$

The stress function $U(r, \phi, \mu, \varepsilon)$ satisfies the non-linear fourth order partial differential equation:

$$\Delta\Delta U + \frac{\mu\beta^2}{R_0^4}\Big[\frac{1}{r^4} U_\varphi T_\varphi - \frac{1}{r^3}(U_{r\varphi}T_\varphi + U_\varphi T_{r\varphi}) - \\ -\frac{1}{r^2}\left(\frac{U_{rr}T_{\varphi\varphi}}{2} - U_{r\varphi}T_{r\varphi} + \frac{U_{\varphi\varphi}T_{rr}}{2}\right) - \frac{1}{2r}(U_r T_{rr} + U_{rr}T_r) - \\ -\frac{1}{3}\Delta(T\Delta U)\Big] = 0, \tag{4}$$

where Δ is the harmonic operator, while the brackets include the non-linear portion of this equation; T is a known function $U(r, \phi, \mu, \varepsilon)$ of the stress derivative functions; $\beta^2 = \frac{Kg_2^2}{G(3K+G)}$ is a constant characterizing the elastic properties of the physically non-linear plate material; and g_2 is a large dimensionless quantity characterizing the deviation of the assumed non-linear law of elasticity (Ref. 52) form Hooke's law.

Stress function $U(r, \phi, \mu, \varepsilon)$ and displacement components $u(r, \phi, \mu, \varepsilon)$ and $v(r, \phi, \mu, \varepsilon)$ are represented as expansions in terms of the small parameters ε and $\mu = \frac{1}{g_r}$: /22

$$U(r, \varphi, \mu, \varepsilon) = H_0 \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \mu^k \varepsilon^l U^{(k,l)}(r, \varphi), \tag{5}$$

[1] e.g., for square and triangular holes (with rounded corners), this function had the respective form

$$f(\zeta) = \frac{1}{\zeta^3}\left(\text{for } \varepsilon = \frac{1}{9} \div \frac{1}{6}\right) \text{and } f(\zeta) = \frac{1}{\zeta^2}\left(\text{for } \varepsilon = \frac{1}{4}\right).$$

$$u(r,\ \varphi,\ \mu,\ \varepsilon) = H_0 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \mu^k \varepsilon^j u^{(k,\ j)}(r,\ \varphi),$$
$$v(r,\ \varphi,\ \mu,\ \varepsilon) = H_0 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \mu^k \varepsilon^j v^{(k,\ j)}(r,\ \varphi), \qquad (5)$$

where H_0 is a constant characterizing the elastic properties of the plate material. In some non-ferrous metals and their alloys, this value is on the order of 10^5-10^6.

Substituting function $U(r,\ \phi,\ \mu,\ \varepsilon)$ from expression (5) into equation (4) and equating the coefficients of the same powers $\varepsilon^k \mu^j$, we will for every function $U^{(k,j)}(k,j = 1,\ 2,\ 3,\ \ldots,\ n)$ derive an equation

$$\Delta\Delta U^{(k,\ j)}(r,\ \varphi) = L_{k,\ j}(U^{(0,0)},\ U^{(0,1)},\ \ldots,\ U^{(k-1,\ j-1)}), \qquad (6)$$

where L_k is the non-linear operator containing functions $U^{(0,0)}$, $U^{(0,1)}$, $U^{(0,2)}$, ..., $U^{(k-1,\ j-1)}$ of the preceding approximations.

Integrating these equations with the pertinent boundary conditions also representing the solutions as expansions in terms of the powers of the parameters, $\varepsilon^k \mu^j$, we will find* functions $U^{(k,j)}(r,\ \phi)$. By the *nth* approximation, we mean the function

$$U_n(r,\ \varphi) = H_0 \sum_{k,\ j=0}^{k+j=n} \mu^k \varepsilon^j U^{(k,\ j)}(r,\ \varphi). \qquad (7)$$

Let us examine the simplest example taken from (Ref. 29) involving unidirectional, uniform extension to infinity by forces p = const of a physically non-linear plate with the above indicated non-linearity (Ref. 52) and an elliptical hole with semiaxes a and b (Figure 5).

The stresses σ_θ on the edge of the hole, which are found with an accuracy of the second approximation, are

$$\sigma_\theta = 2p\,[1 - 1{,}500\lambda p^2 + 10{,}605\lambda^2 p^4 + + 2\varepsilon(\cos 2\theta + \varepsilon \cos 4\theta - 5{,}33\lambda p^2 \cos 2\theta)]. \qquad (8)$$

From expression (8) we see that coefficient $K = \frac{\sigma_\theta}{p}$ of stress concentration in the given version of the physical non-linearity of the plate material depends *non-linearly* both on the magnitude of tensile forces p (Fig. 5), on parameter $\lambda = \mu\beta^2$ characterizing the elastic properties of the plate material, and on the ellipticity of the hole characterized by the parameter $\varepsilon = \frac{a-b}{a+b}$. /23

In expression (8), after setting ε equal to zero, we derive the value of k for a round hole, and when $\lambda = 0$, the value of k for an elliptical hole if the plate material follows Hooke's law. In the latter case, however, there is an exact solution to this problem (Ref. 77). Comparing the corresponding values of k from the exact and approximate solution to expression (8) when

* The components of the stressed and deformed state in the curvilinear system of coordinates $(\rho,\ \theta)$ are determined just as in (Ref. 30 and 102).

$\lambda = 0$, we obtain a clear idea of the rate of convergence of the approximate method of solution proposed by Guz' et al. (Ref. 29).

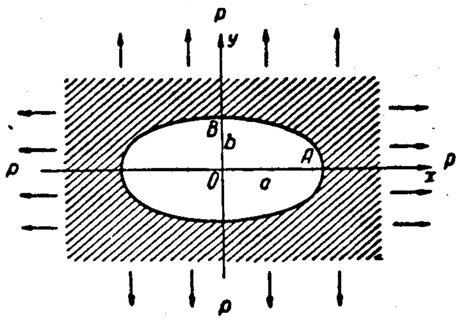


Figure 5

The table gives such a comparison. From the data presented in this table, we see that the second approximation even for a greatly elongated ellipse ($\frac{a}{b} = 1.6$) gives a k value which agrees very well with the exact value (no more than 3.0% discrepancy). The table gives k values at two points (see Fig. 5): A (numerator) and B (denominator) for various values of p, $\frac{a}{b}$, and different materials -- aluminum bronze ($\lambda_1 = 0.055 \cdot 10^{-6} \frac{1}{\text{bar}^2}$) and open-hearth steel ($\lambda_2 = 0.033 \cdot 10^{-6} \frac{1}{\text{bar}^2}$).

It is of great theoretical and practical interest to study the stress states near holes in plates and shells when they are supercritically and plastically deformed, but there are extremely few such studies.

Ya. F. Kayuk (Refs. 53-55) examines the stress state near a circular hole when a plate is bent with large deflections which a plate undergoes when it loses its stability and passes into the region of post-critical elastic deformations. Von Karman's equations underlie the investigations, and the axisymmetrical case of plate buckling is examined. The equations are solved by the small parameter method with improved convergence. The results obtained indicate that -- if the internal contour of the hole is free and the external one is rigidly restrained -- then, as the load on the internal contour becomes larger, the stress concentration also becomes larger because of increased annular force and moment.

In case the external contour is hinged to the internal one there is a rise /25 in stress concentration because of the increase in annular moments and, a decrease in this concentration because of annular forces.

Shells weakened by holes. The first work studying the stress state around holes in shells was A. I. Lur'ye's work published in 1946 (Ref. 69) in which he proposed an approximate method of determining the stress state on the boundary of a small circular hole in the lateral surface of a circular cylindrical shell. This work served as the point of departure for Yu. A. Shevlyakov, I. M. Pirogov, and N. P. Fleishman, who examined a number of interesting and pragmatically important problems for a cylindrical shell weakened by a small round hole under different external loads and with different hole reinforcements. It should be noted that the solutions obtained by A. I. Lur'ye's method permit the stress state to be determined only on the boundary of a circular hole, and under the condition that this hole be of small size.

VALUE OF K

a/b	(a) Линейная теория		(d) Нелинейная теория							
			p=600		p=800		p=1000		p=1200	
	(b) Точное решение	(c) Приближенное решение	(e) Бронза	(f) Сталь	(e) Бронза	(f) Сталь	(e) Бронза	(f) Сталь	(e) Бронза	(f) Сталь
1,00	2,000 / 2,000	2,000 / 2,000	1,960 / 1,960	1,968 / 1,968	1,922 / 1,922	1,947 / 1,947	1,900 / 1,900	1,925 / 1,925	1,894 / 1,894	1,906 / 1,906
1,05	2,101 / 1,905	2,097 / 1,905	2,040 / 1,865	2,062 / 1,879	2,005 / 1,845	2,036 / 1,862	1,973 / 1,833	2,009 / 1,847	1,955 / 1,839	1,983 / 1,835
1,10	2,212 / 1,818	2,198 / 1,819	2,130 / 1,788	2,156 / 1,798	2,087 / 1,775	2,126 / 1,786	2,046 / 1,773	2,093 / 1,777	2,016 / 1,790	2,060 / 1,772
1,20	2,444 / 1,667	2,435 / 1,669	2,310 / 1,657	2,342 / 1,660	2,253 / 1,657	2,304 / 1,656	2,194 / 1,672	2,260 / 1,657	2,143 / 1,712	2,214 / 1,665
1,30	2,516 / 1,539	2,587 / 1,546	2,487 / 1,550	2,526 / 1,546	2,419 / 1,561	2,491 / 1,550	2,343 / 1,594	2,426 / 1,561	2,272 / 1,653	2,368 / 1,581
1,50	3,000 / 1,333	2,960 / 1,360	2,829 / 1,391	2,879 / 1,377	2,738 / 1,427	2,820 / 1,394	2,634 / 1,486	2,749 / 1,422	2,529 / 1,580	2,670 / 1,463
1,60	3,200 / 1,250	3,136 / 1,289	2,898 / 1,427	2,990 / 1,371	2,724 / 1,546	2,881 / 1,438	2,515 / 1,711	2,746 / 1,530	2,227 / 1,935	2,589 / 1,649

(a) - Linear theory; (b) - precise solution; (c) - approximate solution; (d) - nonlinear theory; (e) - bronze; (f) - steel.

/24

A survey of the papers published up to November, 1961, on the use of variational methods, finite difference methods, and various experimental methods employed in studying stress concentration near holes in shells may be found in the author's paper (Ref. 98).

Let us discuss papers which have appeared recently.

I. M. Pirogov (Refs. 86-90) has continued with Lur'ye's method to investigate the stress state in a cylindrical shell around a small circular hole. He has studied (Refs. 86, 87) the effect of reinforcement ring rigidity and (Ref. 89) the effect of press-fitting an elastic ring in the stressed state at the boundary of a small circular hole. From these and preceding works by I. M. Pirogov, certain conclusions may be drawn as to the stress distribution on the boundary of a small circular hole in a cylindrical shell. These conclusions are as follows:

1. If the basic stress state of the shell (panel) is determined by components of the momentless group, the stress state on the boundary of the hole at its most dangerous points is determined by the forces of this group. The effect of flexure stresses at those boundary points may be neglected in the case of free holes; in the case of reinforced holes, the effect of flexure stresses must be taken into consideration.

2. If the basic stress state in the shell is determined by components of the *moment group*, these components will also be fundamental in determining the stress state on the boundary of the hole at points of maximum stress. In the case of free holes, the stresses may be neglected which are uniformly distributed over the thickness of the shell; these stresses must be taken into account in reinforced holes.

3. With increased rigidity of the reinforcing ring, the stress concentration *diminishes*, and does so to an increasing degree as the reinforcing ring becomes wider. /26

We would like to note that the first and second conclusions result from the fact that by A. I. Lur'ye's method a correction is introduced to the plane stress state in the first case. In the second, this occurs when the plane plate is bent. This correction is less, respectively, than the plane stress and the stress state in plane plate bending. The third conclusion coincides with the inference derived when studying the corresponding plane problems.

Pirogin (Ref. 88) has presented a complete solution in the polar system of coordinates for a small round hole. In (Ref. 90) he has studied the stress state around a hole in the case where the basic stress state in the shell is determined by hydrostatic pressure.

The author (Ref. 97) has formulated the problem of stress concentration near openings of arbitrary form in shells with *positive and zero*[1] Gaussian

[1] Further experimental research conducted in the Dynamic Strength Laboratory of the Institute of Mechanics, Academy of Sciences, Ukrainian SSR, and in

curvature. He presents boundary conditions and basic equations of the problem in both differential and integral forms (Refs. 98, 176). These works have proposed several approaches to solving problems of stress concentration around curvilinear holes in shells. Common to all these approaches is the fact that the basic equations and boundary conditions of the problems stated are written in a curvilinear system of coordinates, in which one of the coordinate curves on the shell surface coincides with the profile of the hole.

Hole nomenclature -- circular, square, triangular, etc. -- is determined by the type of curve given by the mapping function $\omega(\zeta)$ when ρ = const for the plane variables to which the shell is referred. The basic equation of the problem with boundary conditions and conditions at infinity may also be referred to the same plane. Generally speaking, when we follow this approach, we come to problems with nonseparable variables, and the variables may be separated only for a spherical shell weakened by an elliptical and, in particular, a circular hole, and for a cylindrical shell with a small round hole.

A paper by G. A. Van Fo Fy, V. N. Buyvol, and the author (Ref. 100), studying the stress state in spherical shells weakened by several circular holes proposed a method of successive approximations. This was employed by a later work (Ref. 99) to examine the stress state in a spherical shell weakened by two unequally reinforced holes. From the examples given in (Refs. 99, 100), the conclusion was drawn that in the case of a *spherical shell* weakened by *circular* holes, the hole effect is practically imperceptible at the distance of one radius (between hole profiles) of the larger hole. This conclusion was also confirmed by Buyvol in (Refs. 10-12). /27

It should be noted that because of the difficulties entailed, the authors restricted themselves merely to the first approximation; this afforded no opportunity to investigate reliably the reciprocal effect of holes as they came closer together. Using the method pointed out above, V. N. Buyvol (Refs. 10, 11) in the same (first) approximation studied the stress state in a spherical shell weakened by several symmetrically distributed round holes and by a single eccentric round hole (Ref. 12). In the present author's work (Ref. 98), he has indicated the feasibility of reducing the problem of stress concentration around a curvilinear hole in a shell to a finite-difference problem for a rectangle in a plane with variables ρ, θ. I. O. Guberman (Ref. 27) has pointed out this feasibility for a spherical shell weakened by an elliptical and square hole with rounded corners.

A. N. Guz' proceeded from the problem of stress concentration near curvilinear holes expressed in differential form, which (Ref. 97) may be reduced to finding the complex stress function

$$\Phi(\rho, \theta) = \frac{Eh^2}{\sqrt{12(1-\nu^2)}}\omega(\rho, \theta) + i\varphi(\rho, \theta), \tag{9}$$

from the differential equation

$$\nabla^2\nabla^2\Phi + i\frac{\sqrt{12(1-\nu^2)}}{h}\nabla_k^2\Phi = 0, \tag{10}$$

(1) *(cont.)* the Photoelasticity Laboratory of the T. G. Shevchenko State University in Kiev has demonstrated that the principal system of equations established for shells of positive and zero Gaussian curvature also remains valid for shells of negative Gaussian curvature.

where

$$\nabla^2 = \frac{1}{H^2}\left(\frac{\partial^2}{\partial\rho} + \frac{\partial^2}{\partial\theta^2}\right),$$

$$\nabla_k^2 = \frac{1}{H^2}\left[\frac{\partial}{\partial\rho}\left(\frac{1}{R_\theta}\frac{\partial}{\partial\rho} + \frac{1}{R_{\rho\theta}}\frac{\partial}{\partial\theta}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{R_{\rho\theta}}\frac{\partial}{\partial\theta} + \frac{1}{R_\rho}\frac{\partial}{\partial\theta}\right)\right]$$

under corresponding boundary conditions on the profile of the hole and under conditions at infinity $\rho = \rho_0$. He (Refs. 30, 33, 34) successfully applied the "perturbation theory" method which makes it possible to compute stresses not only along the edge of the hole, but also in the region near it. This method which is conveniently called the "method of boundary shape perturbation", was applied by him to holes for which the mapping function has the form

$$z = \omega(\zeta) = R_0\left[\zeta + \frac{\varepsilon}{\zeta^N}\right], \tag{11}$$

where N is a whole (positive) number.

He extended this approach (Ref. 32) to the case of doubly-connected regions /28
and (Ref. 38) to the case of a cylindrical shell weakened by a small curvilinear hole.

A. N. Guz' and the author (Ref. 102) have extended this "boundary form perturbation" method to holes of arbitrary shape whose profiles have no corner points and whose mapping function looks like equation (3).

The basic system of equations is integrated by the small parameter method. Specifically, all magnitudes (stresses, displacements, given boundary conditions) are represented as a series with respect to the small parameter ε which enters into mapping function (3). Substituting these expansions into both the basic differential equation and the boundary conditions, and comparing the coefficients of terms with like powers of ε, we derive boundary value problems for each of the successive approximations for a shell weakened by a round hole.

Employing this method (Refs. 30, 32-39, 101, 102), A. N. Guz', S. A. Goloborod'ko, and the author have investigated the stress state around the above-mentioned holes in spherical and cylindrical shells.

These papers derived solutions to the following problems with accuracy to terms of the second order, i.e., to terms up to ε^2.

A *spherical* shell under internal pressure and weakened by elliptical (Refs. 30, 101), square (Ref. 36), and triangular holes with rounded corners.

Let us give values of the concentration coefficients $k = \dfrac{T_\theta}{T_\theta^{(0)}}$ for the second approximation for a spherical shell of radius R = 200 cm, R_0 = 10 cm, h = 0.2 cm, ν = 0.3; in the case of an elliptical hole (Ref. 101) (see Figure 5) $\left(\varepsilon = \frac{a-b}{a+b};\ R_0 = \frac{a+b}{2}\right)$.

$$k = 5{,}30 + \varepsilon 19{,}44\cos 2\theta + \varepsilon^2(16{,}93 - 2{,}49\cos 2\theta + 10{,}96\cos 4\theta)],$$
$$k_{max} = 5{,}30 + 19{,}44\varepsilon + 25{,}40\varepsilon^2; \quad (12)$$

in the case of a square hole (Ref. 36) (see Figure 6) $\left[\varepsilon = \frac{1}{9};\ d = R_0(1+\varepsilon)\right]$

$$k = 5{,}85 + 3{,}22\cos 4\theta + 1{,}01\cos 8\theta;$$
$$k_{max} = 10{,}08; \quad (13)$$

in the case of a triangular hole (Figure 7) $\left[\varepsilon = \frac{1}{4};\ d = R_0(1+\varepsilon)\right]$

$$k = 6{,}50 + 6{,}37\cos 3\theta + 2{,}48\cos 6\theta;$$
$$k_{max} = 15{,}35.$$

For a round hole of radius R_0 = 10 cm, the concentration coefficient of the same forces at the hole profile will be /29

$$k = 5.30. \quad (14)$$

From a comparison of the values of k from equations (11) and (12) with its value in expression (14), we see that hole shape strongly affects the value of the coefficient of force concentration.

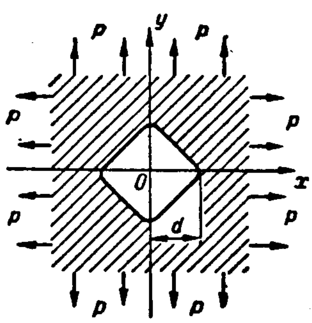


Figure 6

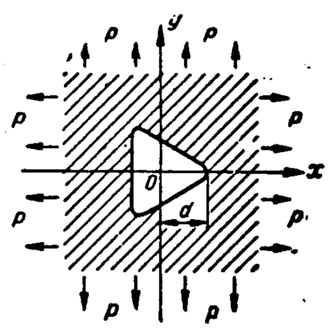


Figure 7

Stress distribution in a *cylindrical* shell weakened by an elliptical hole under uniform internal pressure is given in (Ref. 102), for uniaxial tension (of a panel) in (Ref. 35), and for the same shell weakened by a square hole under uniform internal pressure in (Ref. 39).

Let us discuss the paper by A. N. Guz' (Ref. 40) in which he examines the stress state beside a small round hole in a shell of revolution with a very gently sloping meridian arc.

It is assumed that this shell differs little from the circular cylindrical one, and -- introducing the small parameter characteristic of this deviation -- the author presents the solution in the form of series with respect to this parameter. Just as before, a sequence of boundary value problems is obtained for a small round hole in a circular cylindrical shell. Solution of these problems gives the concentration coefficient at the three most characteristic points $\theta = 0$, $\theta = \frac{\pi}{2}$, situated respectively at the intersection of the Ox, Oy axes (Figure 8) with the hole profile, i.e., at points A, B. Let us give these values.

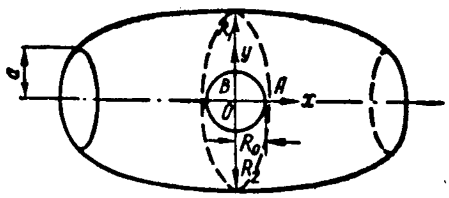


Figure 8

Coefficients of stress concentration around the circular hole in a *doubly curved* shell (Figure 8) are:

For the uniaxial principal stress state (tension) /30

$$k_{\theta=0} = -1 - \frac{\pi\beta^2}{2} - \varepsilon\left(3 + \frac{5}{2}\pi\beta^2\right); \quad (15)$$

$$k_{\theta=\frac{\pi}{2}} = 3 + \frac{\pi\beta^2}{2} + \varepsilon\left(1 + \frac{1}{2}\pi\beta^2\right);$$

$$\varepsilon = \frac{R_2}{R_1};\ \beta = \frac{r_0}{\sqrt{R_2 h}} \cdot \frac{\sqrt[4]{3(1-\nu^2)}}{2}, \quad (16)$$

where R_1 is the radius of shell curvature in the cross section along axis Ox, and R_2 is the same radius in the cross section along the Oy-axis; k is the concentration coefficient of forces T_θ in terms of stresses T_θ^0 in the non-weakened shell at the center of the hole; and $\nu = 0.3$; $\varepsilon = \frac{1}{6}$; $\frac{r_0}{\sqrt{R_2 h}} = 0.6$.

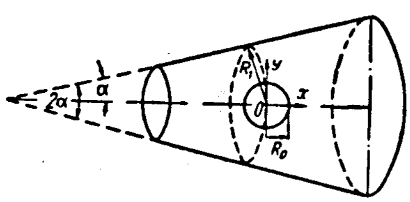


Figure 9

From equations (15) and (16) it follows that under uniaxial tension the value of $k_{\theta=0}$ is 56% greater, and the value of $k_{\theta=\frac{\pi}{2}}$ is 7% greater, as compared with a circular cylindrical shell of radius R_2.

The stress state around a circular hole in a conical shell (Figure 9) under uniaxial tension with $\varepsilon = \frac{R_0}{R_1}\tan\alpha = 0.15$ is characterized by concentration coefficient values of $k_{\theta=0} = -0.85$, $k_{\theta=\pi} = -1.47$.

Computations made for the conical shell (Figure 9) with $\tan\alpha < 1.2$, $R_1 = 100$ cm, $R_0 = 10$ cm, $\varepsilon < 0.12$ $\left(\varepsilon = \frac{R_0}{R_1}\tan\alpha\right)$ demonstrate that k_{max} for this case differs negligibly from its value for a round hole of radius $R_0 = 10$ cm in a cylindrical shell of radius $R_1 = 100$ cm. Moreover, $k = \frac{T_\theta}{T_\theta^{(0)}}$, where T_θ

is the force at the corresponding point of the round hole profile, and $T_\theta^{(0)}$ is the same force in the unweakened conical shell with the same basic stress state at the point where the hole center is located.

Investigation of the stress state near a hole in a conical shell and in a doubly curved shell leads to the following conclusions:

1. The warped state of the meridian in uniaxial tension along the meridian raises k_{max}, but under uniform internal pressure it lowers k_{max}.

2. Shell conicity raises the concentration coefficient k for half the hole profile in the part of the profile which is located closer to the apex of the cone, and lowers the value of k for the other half of the hole profile. /31

Heretofore, in all problems of a cylindrical shell weakened by a curvilinear hole, it was assumed that the hole was small. This resulted from the assumption that $\frac{R_0}{\sqrt{Rh}} \ll 1$.

Recently a solution has been derived (Ref. 103) for the cylindrical shell in the polar coordinate system as a double Fourier series for which the restriction that $\frac{R_0}{\sqrt{Rh}} \ll 1$ is eliminated. This makes it possible to obtain a solution to the above-examined stress concentration problems for *large* holes.

Papers dealing with stress concentration near curvilinear holes in *isotropic* shells have been considered up to this point. For *anistropic* shells, there are no such solutions. Only the single work by A. N. Guz' (Ref. 31) is known, in which Ritz's method is used to study the problem of a stress state around a circular hole reinforced with an absolutely rigid sleeve in an *orthotropic* cylindrical shell under internal hydrostatic pressure.

All the above-considered methods and problems involve investigation of stress concentration around holes in shells under elastic deformation. There are scarcely any studies of similar problems under elastic-plastic deformation; there is only a restricted number of works on this subject. Thus, (Refs. 17-19) investigate the elastic-plastic problem for a shell of revolution weakened by a round hole whose edge is reinforced by an elastic ring of unlike rigidity. A. A. Il'yushin's method of elastic solutions using the method of finite differences underlies this study. A review (Ref. 98) analyzes these papers; therefore, we will not dwell on them here, but will go on to later works.

Thus, I. Yu. Khoma (Ref. 126), on the basis of equations which he derived (Ref. 124) and of Mises's plasticity condition represented in forces and moments, examines the elastic-plastic problem for a flat spherical shell weakened by a round hole and under internal hydrostatic pressure. The hole is covered with a lid transmitting to its edge only a transverse force of constant magnitude. In this work, it is presupposed that there is no elastic-plastic zone (over the thickness of the shell), since the elastic part of the shell is in direct contact with its plastic portion. The problem is solved by a semi-inverse method using successive approximations.

It is of great interest to continue studies in this direction. The solution of the problem may be continued by introducing an elastic ring in the form of a sleeve of unequal rigidity between shell and cover. The investigation /32
should result in obtaining rigidities of optimum or quasi-optimum reinforcement for a circular hole in a spherical shell.

Based on the above, we may arrive at the conclusion that very little, or almost nothing, has been done in such directions of research as:

1. Stress state in shells of zero and positive, as well as negative, Gaussian curvature around several holes; and reciprical effect of these holes when they are drawn closer together, i.e., in the case of multiply-connected regions.

2. Effect of hole size on the state of shells of zero Gaussian curvature and, in particular, shells of negative Gaussian curvature.

3. Effect of geometrical and physical non-linearity of shell material on stress concentration near curvilinear holes.

4. Stress concentration near narrow holes and slits. Development of a shell slit and crack theory, like the crack theory developed for the plane problem of elasticity theory by G. I. Barenblat, M. Ya. Leonov, V. V. Panasyuk, et al.

5. Application of modern computer technology to solving problems of stress concentration beside holes in shells.

6. Investigation of effects on stress concentration around corner point holes.

Very few works are devoted to experimental stress concentrations around holes both under elastic and under elastic-plastic deformation; to the search for optimum reinforcements for holes in plates, and particularly in shells; as well as to the inverse problem -- i.e., the problem in which, from a given basic stress state in the shell, the shape of the hole must be determined, near which there would be no stress concentration or the coefficient of stress concentration would not exceed a given value.

Temperature problems of stress concentration around holes. At present, greater and greater attention is being devoted to studying temperature stresses in machine and structural elements. A systematic development of the initial equations in the temperature problem of elasticity theory, as well as the solution of certain other problems, may be found in many monographs on this problem. Let us indicate one of them (Ref. 79) which is the most extensive and the most recent.

It is known that a determination of the temperature field must precede a study of the stress state in thermoelastic problems.

Most of the studies of the hole effect on distribution of temperature stresses deal with the case of plane deformation where the temperature is assumed to depend on two coordinates. The solution is found in the same way as when solving an ordinary two-dimensional boundary value problem of thermal conductivity. /33

Investigations of stresses caused by holes (Refs. 2, 114) pertain to problems of this type.

(Ref. 2) thus solves the problem (stationary) for a strip with a thermally insulated round hole under the condition that the edges of the strip have constant, but different, temperatures. (Ref. 76) investigates temperature stresses in the proximity of an infinite sequence of round holes in a plate with uniform heat flux. It is assumed that the plate is thermally insulated on the edges of the holes. (Ref. 122) considers the solution to the temperature problem where the temperature distribution is given in the form of trigonometric series in an infinite plate with an elliptical hole. (Ref. 155) examines the solution to the temperature problem for a medium with a spherical or cylindrical cavity with a given uniform heat flux at infinity. Distribution of temperatures and of the stresses caused by them is computed. (Ref. 156) finds temperature distribution in a plate with a thermally insulated oval hole under the effect of a uniform heat flux. The exterior of the hole is mapped onto the exterior of a unit circle by the function

$$\omega(\zeta) = a\zeta + \frac{b}{\zeta} + \frac{c}{\zeta^2}.$$

(Ref. 49) is a generalization of (Ref. 156) for the case where the exterior of the hole is mapped onto the exterior of a unit circle by the function

$$\omega(\zeta) = a_1\zeta + \frac{a_{-1}}{\zeta} + \frac{a_{-2}}{\zeta^2} + \ldots + \frac{a_{-n}}{\zeta^n}.$$

(Ref. 165) analyzes the effect of a plane heat source which is periodic in time and which encounters a cylindrical or spherical cavity impenetrable to heat and load-free in an infinite elastic body.

(Refs. 70 and 171) inquire into questions of stress around holes with given temperature values on the edges of these holes. Thus, (Ref. 70) solves the problem for stationary temperature distribution in the case where the heat fields for a group of round holes are identical, and the temperature along the edges of these holes is the same -- more exactly, constant. (Ref. 171) examines the solution to the stationary temperature problem for an infinite plane with two round holes of the same radius whose boundaries are kept at temperatures of (+T) and (-T).

The results obtained in the above papers are directly transferred to the case of a generalized plane stress state, only on condition that the surfaces of the plates are heat insulated, since computing heat emission from the plate surfaces essentially alters the problem of thermal conductivity. /34

There are almost no publications investigating stresses around holes for plates and shells weakened by holes taking into account heat emission from their surfaces either for stationary or for unstationary regimes. For the case of bending of plates with holes (Ref. 71 and 157) have solved only a few problems under the condition that the temperature varies across the thickness of the slab, remaining unchanged in its central plane. (Ref. 71) investigates thermal stresses in an elastic plate with an infinite number of symmetrically arranged round holes where temperature varies over plate thickness. (Ref. 157) presents formulas and graphs for deflections, moments, and transverse forces in circular plates with round holes with a linear temperature gradient over the thickness and different boundary conditions assigned to the edges.

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3 THEORY OF EQUILIBRIUM CRACKS IN AN ELASTIC LAYER 6

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Let us investigate the plane and three-dimensional problems of stable cracks in a halfspace and in a layer having the thickness 2 h. /39

Stable cracks in an elastic band. Let a longitudinal stable crack having the length 2a -- where a is the diameter halflength of the region Ω occupied by the crack in the plane --arise in an infinite, elastic band having the width 2h under the influence of the pressure q(x) which is variable over the length and which pushes the crack apart.

We shall assume that the crack is located symmetrically with respect to the band edges, and the relative band thickness which is determined by the dimensionless parameter $\lambda = \frac{h}{a}$ is quite large. We must determine the form of the crack $\gamma(x)$ and its dimensions for a given pressure q(x).

The boundary conditions of the problem on the band axis of symmetry y = 0 have the following form

$$\tau_{xy} = 0;\ v = 0 \text{ for } |x| > a;\ \sigma_y = q(x) \text{ for } |x| < a. \qquad (1)$$

The boundary conditions on the band edges y = $\pm$ h may be as follows:

(1) The band is squeezed between two absolutely rigid bases; there is no friction force between the bases and the band:

$$\tau_{xy} = 0;\ v = 0;$$

(2) The band is squeezed between two absolutely rigid bases; there is complete adhesion between the bases and band:

$$u = 0;\ v = 0; \qquad (3)$$

(3) The band edges are free of stress: /40

$$\tau_{xy} = 0;\ \sigma_y = 0. \qquad (4)$$

All three problems of the theory of elasticity may be reduced by operational calculus methods determining the function $\gamma(x)$ characterizing the form of the crack, from the following integral equation;[1]

$$\int_{-a}^{a} \gamma(\xi) M\left(\frac{\xi - x}{h}\right) d\xi = \frac{\pi h^2}{\Delta} q(x) \quad (|x| < a), \qquad (5)$$

where

$$\Delta = \frac{E}{2(1-\sigma^2)};$$

[1] The kernel M(t) is used in the sense of generalized functions.

where E and σ are the elastic band constants;

$$M(t) = -\int_0^\infty uL(u)\cos(ut)\,du \quad \left(t = \frac{\xi - x}{h}\right). \tag{6}$$

the functions L(u) for the problems under consideration have the following form

$$1)\ L(u) = \frac{\operatorname{sh} 2u + 2u}{\operatorname{ch} 2u - 1}; \tag{7}$$

$$2)\ L(u) = \frac{2\varkappa \operatorname{ch} 2u + \varkappa^2 + 1 + 4u^2}{2\varkappa \operatorname{sh} 2u - 4u} \quad (\varkappa = 3 - 4\sigma); \tag{8}$$

$$3)\ L(u) = 2\frac{\operatorname{sh}^2 u - u^2}{\operatorname{sh} 2u + 2u}. \tag{9}$$

As may be readily noted, they all have the following property:

$$L(u) \to 1 + O(e^{-2u}) \text{ for } u \to \infty. \tag{10}$$

It is known that one requisite condition for the existence of stable cracks is the stipulation that the function $\gamma(x)$ and its first derivative vanish at the points $x = \pm a$ (Ref. 3, 4). This provides for smooth closing of the crack edges and the finite nature of stress at its edges.

Thus, we must find the solution of equation (5) under the following conditions

$$\gamma(\pm a) = \gamma'(\pm a) = 0. \tag{11}$$

Let us find the function K(t) which satisfies the following equation:

$$K''_t = M(t). \tag{12}$$

Within an accuracy of the linear term, we obtain /41

$$K(t) = -\ln|t| + \int_0^\infty [1 - L(u)]\left[1 - \cos(ut)\frac{du}{u}\right]. \tag{13}$$

Let us now rewrite equation (5), employing (12), in the following form

$$\int_{-a}^{a} \gamma(\xi)\frac{d^2}{d\xi^2}K\left(\frac{\xi - x}{h}\right)d\xi = \frac{\pi}{\Delta}q(x)\ (|x| < a). \tag{14}$$

Integrating by parts twice and taking (11) into account, we may reduce the problem to solving the following integral differential equation:

$$\int_{-a}^{a} \gamma''(\xi)K\left(\frac{\xi - x}{h}\right)d\xi = \frac{\pi}{\Delta}q(x)\ \ (|x| < a) \tag{15}$$

under the boundary conditions (11).

For large values of the parameter λ (which corresponds to small t) the kernel K(t) of equation (15) may be represented as follows

$$K(t) = -\ln|t| + \sum_{i=1}^{\infty} a_i t^{2i} \quad \left(0 < t < \frac{2}{\lambda}\right). \tag{16}$$

where the constants a_i are determined by the following relationships

$$a_i = \frac{(-1)^{i+1}}{(2i)!}\int_0^\infty u^{2i-1}\,[1 - L(u)]\,du. \tag{17}$$

It may be shown that the series in formula (16) has a convergence radius $\rho = 2$, from which it follows that all of the results based on formula (16) may be reliably employed in the case of $\lambda > 1$.

The constants a_i $(i = 1,2)$ determined by numerical integration have the following form for the problems under consideration

$$\begin{aligned} &1)\ a_1 = -1{,}233;\ a_2 = 0{,}172; \\ &2)\ a_1 = -1{,}738;\ a_2 = 0{,}312; \qquad (\sigma = 0{,}3); \\ &3)\ a_1 = 1{,}143;\ a_2 = -0{,}429. \end{aligned}$$

Let us now make a detailed investigation of the case when $q(x) = q - Ax^2$. We find $\gamma''(x)$ from equation (15), with a kernel of the type (16), according to formulas (2.9) and (2.10) given in (Ref. 1). Then, integrating the expression obtained twice over x and satisfying the boundary condition (11), we may determine the following within an accuracy of $\frac{1}{\lambda^4}$

$$\gamma(x) = -\frac{2q}{\Delta a^2}\left(\frac{1}{3} - \frac{a^2}{2\lambda^4}\right)\left(1 - \frac{a_1}{2\lambda^2} - \frac{2a_2}{\lambda^4}\right)^{-1}(a^2 - x^2)^{3/2} \tag{18}$$

under the additional condition /42

$$q = \frac{Aa^2}{2}\left(1 - \frac{a_1}{2\lambda^2} - \frac{2a_2}{\lambda^4}\right). \tag{19}$$

Condition (19) serves to determine the halflength of the cracks a.

Let us find the expression for the total stress acting upon a crack edge:

$$P = \int_{-a}^{a} q(x)\,dx = 2qa\left[\left(1 - \frac{a_1}{2\lambda^2} - \frac{2a_2}{\lambda^4}\right)^{-1}\right]. \tag{20}$$

The numerical utilization of formulas (18) - (20) in the case of $\lambda \geqq 2$ reveals the following. With a decrease in the band thickness h and for unchanged parameters of the pressure q and A, in the first and second problems there is a decrease in the crack dimensions (longitudinal and transverse), the zone of negative pressures at the ends of the crack -- which are requisite for maintaining the crack in an equilibrium state -- decreases, and the total stress at the crack edge increases. In the case of the third problem, the situation is different. The crack dimensions increase, the zone of negative pressures at the crack ends increases, and the total stress P decreases.

Stable cracks in an elastic layer. Investigating the problems given above in the three-dimensional version, we are led to determine the function $\gamma(x, y)$ characterizing the crack form from the following integral equation:[1]

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[1] The kernel M(t) is used in the sense of generalized functions.

$$\iint_{\Omega} \gamma(\xi, h)\, M\left(\frac{R}{h}\right) d\xi d\eta = \frac{2\pi h^3}{\Delta} q(x, y) \quad (x, y) \in \Omega, \tag{21}$$

where Ω is the region occupied by the crack in the plane,while $a = \frac{1}{2} \max_{\Omega} R$, $R = \sqrt{(\xi - x) + (\eta - y^2)}$, $q(x, y)$ is the pressure pushing the crack apart;

$$M(t) = \int_0^{\infty} u^3 L(u) J_0(ut)\, du \left(t = \frac{R}{h}\right); \tag{22}$$

J -- Bessel function of zero order; the functions L(u) have the form (7) - (9).

In the case of stable cracks, we must find the solution of (21) which satisfies the following condition (Ref. 3, 4):

$$\gamma(x, y) = \frac{\partial}{\partial n} \gamma(x, y) \tag{23}$$

on the contour L of the region Ω

Let us find the function K(t) which satisfies the equation /43

$$h^2 \nabla^2_{\xi,\eta} K(t) = M(t). \tag{24}$$

Within an accuracy of the harmonic function, we obtain

$$K(t) = \frac{1}{t} + \int_0^{\infty} [1 - L(u)][1 - J_0(ut)]\, du. \tag{25}$$

Let us now rewrite equation (21), employing (25), in the following form

$$\iint_{\Omega} \gamma(\xi, \eta)\, \nabla^2_{\xi,\eta} K\left(\frac{R}{h}\right) d\xi d\eta = \frac{2\pi h}{\Delta} q(x, y); \ (x, y) \in \Omega. \tag{26}$$

Integrating by parts and taking (23) into account, we may reduce the problem to solving the following integral-differential equation:

$$\iint_{\Omega} \nabla^2 \gamma(\xi, \eta)\, K\left(\frac{R}{h}\right) d\xi d\eta = \frac{2\pi h}{\Delta} q(x, y); \ (x, y) \in \Omega. \tag{27}$$

For the case $\lambda = \frac{h}{a} = \infty$, equation (27) assumes the following form

$$\iint_{\Omega} \nabla^2 \gamma(\xi, \eta) \frac{d\xi d\eta}{R} = \frac{2\pi}{\Delta} q(x, y); \ (x, y) \in \Omega. \tag{28}$$

By way of an example, let us present the solution of the problem of an elliptic stable crack in an elastic halfspace, obtained by solving equation (28) under the boundary conditions (23).

Let the pressure q(x, y) have the following form

$$q(x, y) = q - A\frac{x^2}{a^2} - B\frac{y^2}{b^2}. \tag{29}$$

Then the function $\gamma(x, y)$ may be determined by the formula

$$\gamma(x, y) = \frac{2bq}{3\Delta E(e)}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{3/2}, \tag{30}$$

and the following two relationships may be fulfilled:

$$A = \frac{q}{e^2 E(e)}\{e^2 E(e) + [K(e) - E(e)](1 - e^2)\}; \tag{31}$$

$$B = \frac{q}{e^2 E(e)}\{2e^2 E(e) - [K(e) - E(e)](1 - e^2)\}, \tag{32}$$

which must be regarded as conditions determining the value of the semiaxes a and b of the elliptic crack. In formulas (30) - (32), K(e) and E(e) are the complete elliptic integrals, and e is the eccentricity.

The total stress acting upon a crack edge is given by the formula /44

$$P = \iint_{\Omega} q(\xi, \eta)\, d\xi\, d\eta = \frac{\pi}{4} abq. \tag{33}$$

Axisymmetric case. For large values of the parameter λ (which corresponds to small p), the kernel K(t) of equation (27) may be represented in the following form

$$K(t) = \frac{1}{t} - \sum_{i=1}^{\infty} a_i t^{2i} \quad \left(0 < t < \frac{2}{\lambda}\right), \tag{34}$$

where the constants a_i are determined by the relationships

$$a_i = \frac{(-1)^i}{(2i!)^2}\int_0^{\infty} [1 - L(u)]\, u^{2i}\, du. \tag{35}$$

It may be shown that the series in formula (34) have the convergence radius $\rho = 2$, from which it follows that all the results based on formula (34) may be utilized reliably in the case of $\lambda > 1$.

The constants a_i (i = 1) determined by numerical integration are as follows for the problems under consideration:

$$1)\ a_1 = 0{,}603;\ 2)\ a_1 = 0{,}971;\ 3)\ a_1 = -1{,}067. \tag{36}$$

Let us now make a detailed investigation of the case of axisymmetric stable cracks in an elastic layer, pushed apart by the pressure

$$q(r) = q - Ar^2. \tag{37}$$

We obtain $\nabla^2\gamma(x, y)$ from equation (26) with a kernel of the form (34), following the method advanced in (Ref. 2). Solving the Poisson equation obtained under the boundary conditions (23), we may determine the following with an accuracy of $\frac{1}{\lambda^4}$

$$\gamma(x, y) = \frac{4qa}{3\Delta\pi}\left(1 - \frac{16a_1}{15\pi\lambda^3}\right)\left(1 - \frac{r^2}{a^2}\right)^{3/2}, \tag{38}$$

under the additional condition

$$A = \frac{3q}{2a^2}\left(1 - \frac{16a_1}{15\pi\lambda^3}\right). \qquad (39)$$

Condition (39) serves to determine the radius of the crack a.

The total stress acting on the crack edge has the form

$$P = \frac{\pi a^2 q}{4}\left(1 + \frac{16a_1}{5\pi\lambda^3}\right). \qquad (40)$$

A numerical study of formulas (38) - (40) in the case of $\lambda \geqq 2$ shows that, /45
when the thickness of the layer h decreases, the qualtative picture of the phenomenon being studied fully coincides with the numerical utilization of formulas (18) - (20).

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LIMITING EQUILIBRIUM OF A PLATE WEAKENED BY A SYSTEM OF CRACKS SITUATED ALONG A STRAIGHT LINE AT AN ANGLE WITH RESPECT TO THE DIRECTION OF TENSILE FORCES

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In order to derive a more comprehensive (general) picture of the influence /46
of defects such as cracks upon the supporting power of brittle bodies, we must study the law governing the propagation of cracks (both rectangular and curvilinear cracks) in the general case of the stress state of a body with cracks -- particularly in the case of an arbitrary crack orientation in the field of tensile stress.

This report investigates one of the simplest problems of this type: the problem of the limiting equilibrium state of an infinite elastic plate weakened by rectilinear cracks located along the same line, which is directed at a certain angle toward the line of tension.

We may assume that the system of rectilinear cracks (a_j, b_j), where $j = 1, 2, \ldots, n$, in an infinite elastic plate is located along the transverse axis (see the figure). Let us assume that the crack edges are free of external stress, and that uniformly distributed stress (increasing monotonically) p is directed at the angle α to the x-axis at infinitely removed points. For this problem, let us determine the limiting (critical) values of the stress $p = p_*^{(\lambda)}$; when these values are reached, at the end with the abscissa λ (where λ is any of the abscissas a_j, b_j) the crack reaches a state of mobile equilibrium (it begins to propagate).

As was shown in (Ref. 1), the external stress applied to a body with a macroscopic crack will be a limiting stress, if the stress produced by it in the vicinity of the crack ends -- which may be calculated without allowance
for cohesion -- has the singularity $K/\pi\sqrt{r}$, where K is the cohesion modulus, /47
and r is a small distance from the crack end.

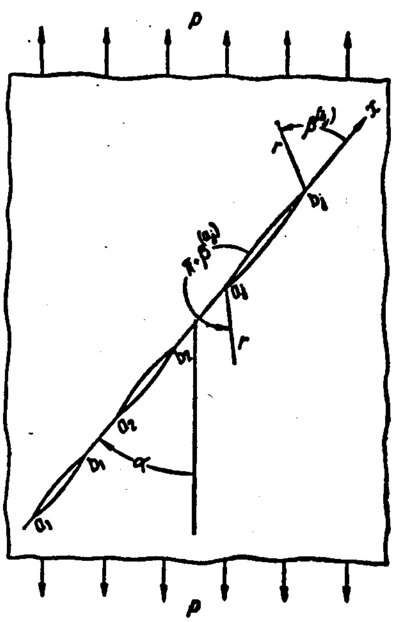
In addition to this condition, we may employ the following assumption for the effective determination of the limiting stress -- as was done in (Ref. 6). This assumption stipulates that the initial propagation direction of an arbitrarily oriented, rectilinear (or curvilinear) crack coincides with the direction in which the normal tensile stress reaches a maximum intensity.

Thus, in order to determine the limiting values of the external stress $p = p = p_*^{(\lambda)}$, we obtain the following relationships

$$\lim_{r\to 0}\{\sqrt{r}\,\sigma_\beta^{*(\lambda)}(r, \beta_*^{(\lambda)})\} = \frac{K}{\pi}; \tag{1}$$

$$\lim_{r\to 0}\left\{\sqrt{r}\,\frac{\partial\sigma_\beta^{(\lambda)}(r, \beta^{(\lambda)})}{\partial\beta}\right\}_{\beta^{(\lambda)} = \beta_*^{(\lambda)}} = 0, \tag{2}$$

where r, $\beta^{(\lambda)}$ are the polar coordinates with the origin at the apex of the



crack with the abscissa λ and the polar axis directed along the positive direction of the x-axis (see the figure); $\sigma_\beta^{(\lambda)}(r, \beta^{(\lambda)})$ -- elastic tensile stress perpendicular to the plane $\beta^{(\lambda)} = \text{const}$; $\beta^{(\lambda)}$ -- polar angle characterizing the direction of the initial crack propagation at the end with the abscissa λ; $\sigma_\beta^{*(\lambda)}(r, \beta_*^{(\lambda)})$ -- stress values $\sigma_\beta^{(\lambda)}(r, \beta)$ in the case of $p = p_*^{(\lambda)}$.

Based on the results given in (Ref. 3, 6, 12), the stress $\sigma_\beta^{(\lambda)}$ in the polar coordinate system $(r, \beta^{(\lambda)})$ in the vicinity of the crack end with the abscissa λ may be written in the following form:

$$\sigma_\beta^{(\lambda)} = \frac{1}{4\sqrt{2r}}\left\{k_1^{(\lambda)}\left[3\cos\frac{\beta^{(\lambda)}}{2} + \cos\frac{3}{2}\beta^{(\lambda)}\right] - 3k_2^{(\lambda)}\left[\sin\frac{\beta^{(\lambda)}}{2} + \sin\frac{3}{2}\beta^{(\lambda)}\right]\right\} + 4A^{(\lambda)}\sin^2\beta^{(\lambda)} + O(r^{\frac{1}{2}}), \qquad (3)$$

where λ is any of the abscissas a_j, b_j.

The quantities $k_1^{(\lambda)}$ and $k_2^{(\lambda)}$, which depend on the form of the loading and the parameters determining the crack configuration, are usually called (Ref. 1, 7) the intensity coefficients (or concentration coefficients) of the stress at the crack apex λ in the case of symmetrical $k_1^{(\lambda)}$ and asymmetrical /48 $k_2^{(\lambda)}$ stress distribution, respectively.

If a procedure similar to that presented in (Ref. 7) is followed when calculating these coefficients, we obtain the following formula which relates the intensity coefficients to the Muskhelishvili function $\Phi_{(z)}$, which describes the problem under consideration:

$$k^{(\lambda)} = k_1^{(\lambda)} - ik_2^{(\lambda)} = 2\sqrt{2}\lim_{z\to\lambda}(z-\lambda)^{\frac{1}{2}}\Phi(z). \qquad (4)$$

The problem of the stress-deformed state of an infinite plate with rectilinear cracks located along one line was studied in (Ref. 3, 8, 9, 11). In particular, according to (Ref. 3), for the problem under consideration the function $\Phi(z)$ by means of which the intensity coefficients are determined has the following form

$$\Phi(z) = \frac{P_n(z)}{X(z)} + \frac{p}{4} e^{2ie}, \tag{5}$$

where $P_n(z) = C_0 z^n + C_1 z^{n-1} + \ldots + C_n$, while $C_0 = \frac{p}{4}(1 - e^{2ie})$;

$$X(z) = \prod_{\kappa=1}^{n} (z - a_\kappa)^{\frac{1}{2}} (z - b_\kappa)^{\frac{1}{2}}.$$

The term $X(z)$ designates the branch for which $z^{-n}X(z) \to 1$ in the case $z \to \infty$. The coefficients $C_1, C_2, \ldots, C_n$ may be determined from the condition of single-valued displacements (Ref. 3) which leads to the following equation

$$\int_{a_\kappa}^{b_\kappa} \frac{P_n(x)}{X(x)} dx = 0 \quad (\kappa = 1, 2, \ldots, n). \tag{6}$$

Substituting (5) in relationship (4), we find

$$k^{(b_j)} = k_1^{(b_j)} - ik_2^{(b_j)} = 2\sqrt{2} \frac{P_n(b_j)}{(b_j - a_j)^{\frac{1}{2}} \prod_{\kappa=1,\ \kappa\neq j}^{n} (b_j - a_\kappa)^{\frac{1}{2}} (b_j - b_\kappa)^{\frac{1}{2}}}, \tag{7}$$

$$k^{(a_j)} = k_1^{(a_j)} - ik_2^{(a_j)} = -2\sqrt{2} \frac{P_n(a_j)}{(b_j - a_j)^{\frac{1}{2}} \prod_{\kappa=1\ \kappa\neq j}^{n} (a_j - a_\kappa)^{\frac{1}{2}} (a_j - b_\kappa)^{\frac{1}{2}}}. \tag{8}$$

Thus, the intensity coefficients may be determined according to formulas (7) and (8), if only we find the coefficients of the polynomial $P_n(z)$ which must satisfy the system of equations (6). If we now employ expression (3), we may represent equation (1) in the following form /49

$$k_1^{*(\lambda)} \left(3 \cos \frac{\beta_*^{(\lambda)}}{2} + \cos \frac{3}{2} \beta_*^{(\lambda)}\right) - 3k_2^{*(\lambda)} \left(\sin \frac{\beta_*^{(\lambda)}}{2} + \sin \frac{3}{2} \beta_*^{(\lambda)}\right) = \frac{4K\sqrt{2}}{\pi}, \tag{9}$$

where the quantities $k_1^{*(\lambda)}$ and $k_2^{*(\lambda)}$ equal the coefficients $k_1^{(\lambda)}$ and $k_2^{(\lambda)}$, if we set $p = p_*^{(\lambda)}$ in the latter. The angle $\beta_*^{(\lambda)}$ is determined by the relationships (Ref. 6)

$$\beta_*^{(\lambda)} = \pm 2 \arcsin \sqrt{\frac{6n_\lambda^2 + 1 - \sqrt{8n_\lambda^2 + 1}}{2(9n_\lambda^2 + 1)}} \text{ при } k_1^{(\lambda)} > 0; \tag{10}$$

$$\beta_*^{(\lambda)} = \pm 2 \arcsin \sqrt{\frac{6n_\lambda^2 + 1 + \sqrt{8n_\lambda^2 + 1}}{2(9n_\lambda^2 + 1)}} \text{ при } k_1^{(\lambda)} < 0, \tag{11}$$

where

$$n_\lambda = \frac{k_2^{(\lambda)}}{k_1^{(\lambda)}}. \tag{12}$$

The sign "+" corresponds to the value $k_2^{(\lambda)} < 0$, and the sign "-" corresponds to the values $k_2^{(\lambda)} > 0$. It may thus be seen that, by determining the intensity coefficients, we may readily find the limiting loading $p_*^{(\lambda)}$ in each specific case. Let us present the values of the coefficients $k_1^{(\lambda)}$ and $k_2^{(\lambda)}$ for certain special cases of this problem.

1. The case of an infinite plate with three cracks, when

$$a_1 = -c,\ b_1 = -b;\ a_2 = -a,\ b_2 = a;\ a_3 = b,\ b_3 = c.$$

According to formulas (5) - (8), in this case we have

$$\begin{gathered}
k_1^{(c)} - ik_2^{(c)} = p(\sin^2\alpha - i\sin\alpha\cos\alpha)\frac{F(k) - E(k)}{F(k)}\sqrt{c}\sqrt{\frac{c^2 - a^2}{c^2 - b^2}};\\
k_1^{(b)} - ik_2^{(b)} = p(\sin^2\alpha - i\sin\alpha\cos\alpha)\times\\
\times\frac{[(c^2 - a^2)E(k) - (b^2 - a^2)F(k)]}{\sqrt{(b^2 - a^2)(c^2 - b^2)}F(k)}\sqrt{b};\\
k_1^{(a)} - ik_2^{(a)} = p(\sin^2\alpha - i\sin\alpha\cos\alpha)\frac{E(k)}{F(k)}\sqrt{a}\sqrt{\frac{c^2 - a^2}{b^2 - a^2}},
\end{gathered} \tag{13}$$

where F(k) and E(k) are the total elliptic integrals, respectively, of the first and second type with the modulus $k^2 = \frac{c^2 - b^2}{c^2 - a^2}$. On the basis of (12), we may conclude from the latter formulas that

$$n_a = n_b = n_c = \operatorname{ctg}\alpha.$$

2. The case of two equal cracks corresponds to the previous case in the case of $a \to 0$. We find /50

$$\begin{gathered}
k_1^{(b)} - ik_2^{(b)} = p(\sin^2\alpha - i\sin\alpha\cos\alpha)\frac{c^2E(k) - b^2F(k)}{\sqrt{b(c^2 - b^2)}F(k)};\\
k_1^{(c)} - ik_2^{(c)} = p(\sin^2\alpha - i\sin\alpha\cos\alpha)\frac{F(k) - E(k)}{F(k)\sqrt{c^2 - b^2}}c\sqrt{c}.
\end{gathered} \tag{14}$$

Here $k^2 = \frac{c^2 - b^2}{c^2}$.

3. In the case $b \to 0$, we may find the following expression for the intensity coefficients from formula (14) in the case of one crack having the length 2c:

$$k_1 = p\sqrt{c}\sin^2\alpha;\ k_2 = p\sqrt{c}\sin\alpha\cos\alpha. \tag{15}$$

4. In the case of two colinear cracks of unequal length with abscissas of the ends a_1, b_1 and a_2, b_2, the intensity coefficients may be determined by the formulas

$$\begin{gathered}
k_1^{(b_2)} - ik_2^{(b_2)} = 2\sqrt{2}\frac{C_0b_2^2 + C_1b_2 + C_2}{\sqrt{(b_2 - a_2)(b_2 - a_1)(b_2 - b_1)}};\\
k_1^{(b_1)} - ik_2^{(b_1)} = -2\sqrt{2}\frac{C_0b_1^2 + C_1b_1 + C_2}{\sqrt{(b_1 - a_1)(b_2 - b_1)(a_2 - b_1)}};
\end{gathered} \tag{16}$$

$$k_1^{(a_2)} - ik_2^{(a_2)} = -2\sqrt{2}\frac{C_0a_2^2 + C_1a_2 + C_2}{\sqrt{(b_2 - a_2)(a_2 - a_1)(a_2 - b_1)}};$$
$$k_1^{(a_1)} - ik_2^{(a_1)} = 2\sqrt{2}\frac{C_0a_1^2 + C_1a_1 + C_2}{\sqrt{(b_1 - a_1)(a_2 - a_1)(b_2 - a_1)}}, \quad (16)$$

where

$$C_0 = \frac{p}{4}(1 - e^{2i\alpha});$$

$$C_1 = C_0\frac{-(b_2 + a_1)F(k) + 2b_2\prod(m, k) + 2a_1\prod(n, k) + (b_2 - a_1)[I_2(n, k) - I_2(m, k)]}{F(k) - \prod(n, k) - \prod(m, k)};$$

$$C_2 = -C_1\left[a_1 + (b_2 - a_1)\frac{\prod(n, k)}{F(k)}\right] - C_0\left[a_1^2 + 2(b_2 - a_1)a_1\frac{\prod(n, k)}{F(k)} + (b_2 - a_1)^2\frac{I_2(n, k)}{F(k)}\right].$$

Here we have /51

$$I_2(n, k) = \int_0^{\pi/2}\frac{d\varphi}{(1 + n\sin^2\varphi)\sqrt{1 - k^2\sin^2\varphi}},$$

$F(k)$, $\Pi(n, k)$, $\Pi(m, k)$ are the total elliptic integrals of the first and second type with the modulus k and the parameters n and m, while $k^2 = nm$; $n = \frac{b_2 - a_2}{a_2 - a_1}$; $m = \frac{b_1 - a_1}{b_2 - b_1}$. It may be readily seen that

$$n_{a_1} = n_{b_1} = n_{a_2} = n_{b_2} = \operatorname{ctg}\alpha.$$

5. In the case when the plate is weakened by a periodic system of slits having the length 2ℓ and the period 2L, we find the following on the basis of results derived in (Ref. 11, 10, 2):

$$k_1 = p\sin^2\alpha\sqrt{\frac{2L}{\pi}\operatorname{tg}\frac{\pi l}{2L}};\ k_2 = p\sin\alpha\cos\alpha\sqrt{\frac{2L}{\pi}\operatorname{tg}\frac{\pi l}{2L}}. \quad (17)$$

As may be seen from expressions (13) - (17), the intensity coefficients may be represented as follows:

$$k_1^{(\lambda)} = p\bar{k}_1^{(\lambda)};\ k_2^{(\lambda)} = p\bar{k}_2^{(\lambda)}. \quad (18)$$

Relationship (9), with allowance for equation (18), may be represented in the following form

$$p_*^{(\lambda)} = \frac{K\sqrt{2}}{\pi}\cdot\frac{1}{\cos^2\frac{\beta_*^{(\lambda)}}{2}\left[\bar{k}_1^{(\lambda)}\cos\frac{\beta_*^{(\lambda)}}{2} - 3\bar{k}_2^{(\lambda)}\sin\frac{\beta_*^{(\lambda)}}{2}\right]}. \quad (19)$$

The angle $\beta_*^{(\lambda)}$, as may be readily seen on the basis of (10), may be expressed by the following formula for the examples presented above

$$\beta_*^{(\lambda)} = -2 \arcsin \sqrt{\frac{6 \operatorname{ctg}^2 \alpha + 1 - \sqrt{8 \operatorname{ctg}^2 \alpha + 1}}{2 (9 \operatorname{ctg}^2 \alpha + 1)}} \quad (20)$$

Thus, the initial propagation direction of the cracks depends only on the orientation of the line (the angle α), along which the cracks are located, and does not depend on the coordinates of their ends.

We should note that the problem of the boundary conditions for a plate with cracks located along a line perpendicular to the plate loading direction, was investigated in (Ref. 2, 4, 5). The results presented in these articles are obtained from formulas (13) - (20) in the case $\alpha = \frac{\pi}{2}$.

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BENDING OF A THIN ISOTROPIC PLATE WITH A HOLE OF GENERAL FORM, TAKING TEMPERATURE STRESSES INTO ACCOUNT

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Basic relationships. Let us investigate elastic equilibrium of an isotropic plate having the constant thickness h. Let the plate be located in a stationary, thermal field with a temperature which changes according to a linear law over the thickness /53

$$T = zt(x, y). \tag{1}$$

We shall assume that the temperature along the upper surface is $T_1(x, y)$, and that it is $T_2(x, y)$ along the lower surface. Under these conditions, the middle plane of the plate is not flat. During bending, bending moments and torques occur, which may be determined according to the following formulas given in (Ref. 4), with allowance for temperature

$$\left.\begin{aligned} M_x &= -D\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} + \alpha(1+\nu)t\right], \\ M_y &= -D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2} + \alpha(1+\nu)t\right], \\ H_{xy} &= -D(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y}. \end{aligned}\right\} \tag{2}$$

Here $D = \frac{Eh^3}{12(1-\nu^2)}$ is cylindrical rigidity; E, ν -- Young's modulus and Poisson coefficient; α -- coefficient of linear expansion; t(x, y) -- temperature; w(x, y) -- bending of the plate, which satisfies the following equation (Ref. 4):

$$\nabla^2\nabla^2 w = -\alpha(1+\nu)\nabla^2 t; \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{3}$$

The general solution of equation (3) may be written in the following form /54

$$w = w_0 + \operatorname{Re}[\bar{z}\varphi(z) + \chi(z)], \tag{4}$$

where $w_0(x, y)$ is a particular solution of the nonhomogeneous equation (3), which depends on temperature t(x, y); $\bar{z} = x - iy$; $\phi(z)$ and $\chi(z)$ -- analytical functions.

The bending moments and the torques, as well as the intersection forces, are determined from the following formulas (Ref. 5)

$$\left.\begin{aligned} &M_y + M_x = M_y^0 + M_x^0 - 2D(1-\nu)[\varphi'(z) + \overline{\varphi'(z)}], \\ &M_y - M_x + 2iH_{xy} = M_y^0 - M_x^0 + 2iH_{xy}^0 + \\ &+ 2D(1-\nu)[\bar{z}\varphi''(z) + \psi'(z)], \\ &N_x - iN_y = N_x^0 - iN_y^0 - 4D\varphi''(z). \end{aligned}\right\} \tag{5}$$

The quantities pertaining to the particular solution and expressed by

$w_0(x, y)$, are given by M_x^0, M_y^0, H_{xy}^0, N_x^0, N_y^0 .

The boundary conditions for the free edge have the following form

$$M_n = M_n^0 + \overline{M}_n = 0; \quad N_n + \frac{\partial H_{nt}}{\partial s} = N_n^0 + \overline{N}_n + \\ + \frac{\partial}{\partial s}(H_{nt}^0 + \overline{H}_{nt}) = 0. \tag{6}$$

If conditions (6) are satisfied along the hole edge, then it may be transformed to the following form (Ref. 7)

$$\varphi'(z) + \varkappa\overline{\varphi'(z)} - e^{2i\alpha}[\bar{z}\varphi''(z) + \psi'(z)] = iC_1 + F(z). \tag{7}$$

Here we have

$$F(z) = \frac{1}{D(1-\nu)}\left[m(s) - i\int_0^s p\,ds\right], \quad \varkappa = -\frac{3+\nu}{1-\nu}, \tag{8}$$

α is the angle between the normal to the contour and the axis Ox.

Let $z = \omega(\zeta)$ be the function which conformally maps the exterior of the circle γ having unit radius onto the exterior of an infinite plate with a hole.

In the mapping, the boundary condition (7) may then be transformed under the circumference condtions γ:

$$\Phi(\sigma) + \varkappa\overline{\Phi(\sigma)} - \frac{\sigma^2}{\overline{\omega'(\sigma)}}[\overline{\omega(\sigma)}\Phi'(\sigma) + \omega'(\sigma)\Psi(\sigma)] = iC_1 + F(\sigma). \tag{9}$$

Here $\sigma = e^{i\nu}$ is the point on the circumference γ; $\Phi(\zeta) = \phi'(\zeta)/\omega'(\zeta)$; $\Psi(\zeta) = \psi(\zeta)/\omega'(\zeta)$ -- holomorphic and single valued functions outside γ, including an infinitely removed point. For $|\zeta| > 1$, in our case they have the following form

$$\Phi(\zeta) = \sum_{k=1}^{\infty} a_k\zeta^{-k}; \quad \Psi(\zeta) = \sum_{k=1}^{\infty} a_k'\zeta^{-k}. \tag{10}$$

If the function $\Phi(\zeta)$ is determined, we may then find the function $\Psi(\zeta)$ in the form given in (Ref. 1), and the following dependence exists between the coefficients /55

$$a_1 = \bar{a}_1'; \quad a_2' = \bar{a}_2', \tag{11}$$

which follows from the conditions of single valued displacements.

The stresses (moments) on the hole contour are found according to the following formulas

$$\left.\begin{aligned} M_\rho &= (1-\nu)F, \\ M_\vartheta &= -(1-\nu)F - 4(1+\nu)D\,\mathrm{Re}\,\Phi(\sigma), \\ H_{\rho\vartheta} &= 4D\,\mathrm{Im}\,\Phi(\sigma). \end{aligned}\right\} \tag{12}$$

General solution of the problem. Let us examine a rectangular, thin, isotropic plate with a hole whose contour is given by equations (Ref. 1)

$$x = R\left(\cos\vartheta + \sum_{k=1}^{n} s_k \cos k\vartheta\right), \quad y = R\left(\sin\vartheta - \sum_{k=1}^{n} s_k \sin k\vartheta\right). \tag{13}$$

Let the plate be located in a stationary thermal field, and we shall assume that t(x, y) = const. In this case, we have the following formulas for the desired quantities depending on temperature

$$\begin{aligned} &w_0 = 0, \quad M_x^0 = M_y^0 = -D(1+\nu)\,\alpha t; \\ &H_{xy}^0 = N_x^0 = N_y^0 = 0. \end{aligned} \tag{14}$$

In the case under consideration $M_n^0 = -D(1 + \nu)\,\alpha t$, $m = -M_n^0$, and then the function F may be determined according to the formula

$$F = \frac{m}{D(1-\nu)},$$

where

$$m = D(1+\nu)\,\alpha t, \quad C_1 = 0, \quad a_1 = 0. \tag{15}$$

Integrating condition (9), as was done in the work (Ref. 1), we obtain the function $\Phi(\zeta)$ in the following form

$$\begin{aligned} \Phi(\zeta) = F\left[\varkappa\left(1 - \sum_{k=1}^{n} k s_k \zeta^{-k-1}\right)\right]^{-1} \Bigl\{ &\sum_{m=2}^{n-1} \zeta^{-m} \sum_{k=m+1}^{n} (m-1)\, s_k \alpha_{k-(m-1)} - \\ &- \sum_{k=1}^{n} k s_k \zeta^{-k-1} \Bigr\}. \end{aligned} \tag{16}$$

The coefficients α_k may be found from the following system

$$\varkappa\left[\alpha_m - \sum_{k=2}^{m-2} (k-1)\, \alpha_{m-k} s_{k-1}\right] = (m-1)\left[\sum_{k=m+1}^{n} s_k \alpha_{k-(m-1)} - s_{m-1}\right], \tag{17}$$

where m takes on the values 2, 3, ..., (n + 1).

Substituting (16) in formulas (12) we obtain /56

$$\left.\begin{aligned} &M_\rho = m, \\ &M_\vartheta = -\frac{m}{3+\nu}\left[3 + \nu + \frac{4(1+\nu)}{L}\left(L_0 + \sum_{k=1}^{n+1} L_k \cos k\vartheta\right)\right], \\ &H_{\rho\vartheta} = -\frac{4}{3+\nu} \cdot \frac{1}{L} \cdot \sum_{k=1}^{n+1} P_k \sin k\vartheta, \end{aligned}\right\} \tag{18}$$

where

$$m = D_t t, \quad D_t = D(1+\nu)\,\alpha; \tag{19}$$

$$\begin{aligned} L = 1 + \sum_{k=1}^{n} k^2 s_k^2 + 2\Bigl\{&\cos\vartheta \sum_{k=2}^{n} k(k-1)\, s_k s_{k-1} - \sum_{k=1}^{n} k s_k \cos(k+1)\vartheta + \\ &+ \sum_{k=3}^{n} \cos(k-1)\vartheta \sum_{j=k}^{n} j\,[j-(k-1)]\, s_j s_{j-(k-1)}\Bigr\}; \end{aligned} \tag{20}$$

$$L_0 = -\sum_{k=1}^{n} k^2 s_k^2 + \sum_{k=3}^{n} \alpha_{k-1} J_1; \tag{21a}$$

$$L_1 = -2\sum_{k=2}^{n} k(k-1)\, s_k s_{k-1} + \sum_{k=3}^{n} \alpha_{k-1} J_2 + \sum_{k=3}^{n-1} \alpha_{k-1} J_3; \tag{21b}$$

$$L_2 = s_1 - J_4 - \sum_{k=3}^{n} s_k \alpha_{k-1} + \sum_{k=3}^{n} \alpha_{k-1} J_5 + \sum_{k=3}^{n-2} \alpha_{k-1} J_6; \tag{21c}$$

$$L_m = (m-1)\,[s_{m-1} - J_7] - (m-1) \sum_{k=m+1}^{n} s_k \alpha_{k-(m-1)} + \sum_{k=3}^{m+1} \alpha_{k-1} J_8 + \sum_{k=3}^{n-m} \alpha_{k-1} J_9 + \sum_{k=1}^{n-(m+1)} \alpha_{k+m} J_{10}; \tag{21d}$$

$$L_n = (n-1)\, s_{n-1}, \quad L_{n+1} = n s_n; \tag{21e}$$

$$P_1 = -\sum_{k=3}^{n} \alpha_{k-1} J_2 + \sum_{k=3}^{n-1} \alpha_{k-1} J_3; \tag{22f}$$

$$P_2 = s_1 - \sum_{k=3}^{n} s_k \alpha_{k-1} - \sum_{k=3}^{n} \alpha_{k-1} J_5 + \sum_{k=3}^{n-1} \alpha_{k-1} J_6; \tag{22g}$$

$$P_m = (m-1)\, s_{m-1} - (m-1) \sum_{k=m+1}^{n} s_k \alpha_{k-(m-1)} - \sum_{k=3}^{m+1} \alpha_{k-1} J_8 + \sum_{k=3}^{n-m} \alpha_{k-1} J_9 - \sum_{k=1}^{n-(m+1)} \alpha_{k+m} J_{10}; \tag{22h}$$

$$P_n = (n-1)\, s_{n-1}, \quad P_{n+1} = n s_n. \tag{22i}$$

Here we have $J_1 = \sum_{j=k}^{n} [j-(k-1)]^2 s_j s_{j-(k-1)}; \quad J_2 = \sum_{j=k}^{n} [j-(k-2)]\,[j-(k-1)]\, s_j s_{j-(k-2)};$ /57

$$J_3 = \sum_{j=k+1}^{n} (j-k)\,[j-(k-1)]\, s_j s_{j-k}; \quad J_4 = 2\sum_{k=3}^{n} k(k-2)\, s_k s_{k-2}; \tag{23a}$$

$$J_5 = \sum_{j=k}^{n} [j-(k-1)]\,[j-(k-3)]\, s_j s_{j-(k-3)}; \tag{23b}$$

$$J_6 = \sum_{j=k+2}^{n} [j-(k-1)]\,[j-(k+1)]\, s_j s_{j-(k+1)}; \tag{23c}$$

$$J_7 = 2\sum_{k=m+1}^{n} k(k-m)\, s_k s_{k-m}; \quad J_8 = \sum_{j=m+1}^{n} j(j-m)\, s_j s_{j-(m-k+1)}; \tag{23d}$$

$$J_9 = \sum_{j=m+k}^{n} [j-(k-1)]\,[j-(k+m-1)]\, s_j s_{j-(k+m-1)}; \tag{23e}$$

$$J_{10} = \sum_{j=k+m+1}^{n} (j-k)\,[j-(k+m)]\, s_j s_{j-k}. \tag{23f}$$

(23g)

The index m takes on the values 3, 4, ..., (n - 1).

Special Holes. Let us investigate a plate whose center has a hole cut out in the form of an ellipse[1] or a regular polygon, whose contours are

[1] The solution was obtained by another method for a plate with an elliptic hole in (Ref. 2). However, it was erroneous, because the result of M.M. Fridman for a circular hole (Ref. 6) does not follow from this solution.

given by the following equations

$$x = R(\cos\vartheta + s_k \cos k\vartheta), \quad y = R(\sin\vartheta - s_k \sin k\vartheta). \qquad (24)$$

In this case, we have the following expressions for the function $\Phi(\zeta)$ and the moments on the hole contour:

$$\Phi(\zeta) = \frac{m}{D(3+\nu)} \cdot \frac{ks_k}{\zeta^{k+1} - ks_k}; \qquad (25)$$

$$M_\vartheta = -m\left\{1 - \frac{4(1+\nu)}{(3+\nu)} \cdot \frac{ks_k[ks_k - \cos(k+1)\vartheta]}{1 + k^2 s_k^2 - 2ks_k \cos(k+1)\vartheta}\right\}; \qquad (26)$$

$$M_\rho = m;$$

$$H_{\rho\vartheta} = -\frac{4m}{3+\nu} \cdot \frac{ks_k \sin(k+1)\vartheta}{1 + k^2 s_k^2 - 2ks_k \cos(k+1)\vartheta}. \qquad (27)$$

The table presents the values of the moments M_ϑ and $H_{\rho\vartheta}$ (in fractions of m) along the contour of an elliptic hole (k = 1), a triangular hole (k = 2), a square hole (k = 3), and a hexagonal hole (k = 5).

We may readily obtain the value of the function $\Phi(\zeta)$ and the moments M_ϑ and H_ϑ for other holes from formulas (16) - (23).

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$\vartheta°$	Отверстие 5							
	Эллиптическое 1 $\left(k=1,\ s_1=\frac{1}{5}\right)$		Треугольное 2 $\left(k=2,\ s_2=\frac{1}{3}\right)$		Квадратное 3 $\left(k=3,\ s_3=\frac{1}{6}\right)$		Шестиугольное 4 $\left(k=5,\ s_5=\frac{1}{15}\right)$	
	M_ϑ/m	$H_{\rho\vartheta}/m$	M_ϑ/m	$H_{\rho\vartheta}/m$	M_ϑ/m	$H_{\rho\vartheta}/m$	M_ϑ/m	$H_{\rho\vartheta}/m$
0	—1,394	0	—4,152	0	—2,576	0	—1,788	0
20	—1,243	—0,212	—0,775	—0,900	—0,761	—0,555	—0,697	—0,242
40	—0,991	—0,246	—0,419	—0,331	—0,482	—0,095	—0,697	0,242
45	—0,939	—0,233	—0,395	—0,239	—0,475	0	—0,842	0,364
60	—0,822	—0,169	—0,370	0	—0,550	0,300	—1,788	0
80	—0,746	—0,059	—0,419	0,331	—1,433	0,805	—0,697	—0,242
90	—0,734	0	—0,515	0,559	—2,576	0	—0,606	0
110	—0,774	0,116	—1,723	1,395	—0,761	—0,419	—1,112	0,242
120	—0,822	0,169	—4,152	0	—0,550	—0,300	—1,788	0
140	—0,991	0,246	—0.775	—0,900	—0,482	0,095	—0,697	—0,242
160	—1,243	0,212	—0,419	—0,331	—0,761	0,555	—0,697	0,242
180	—1,394	0	—0,370	0	—2,576	0	—1,788	0

(1) Elliptic; (2) Triangular; (3) Square; (4) Hexagonal; (5) Hole

If it is assumed in all the formulas that m is the moment which is distributed uniformly along the hole edge, we then obtain the results for a plate with a hole of general form, which is loaded by constant moments m which are distributed uniformly along the hole edge.

The torque $H_{\rho\vartheta}$ assumes maximum values for the openings indicated above in the case of ϑ which may be determined from the equations

$$\cos(k+1)\vartheta = \frac{2ks_k}{1 + k^2 s_k^2}.$$

* * *

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STRESS CONCENTRATION AROUND A HOLE IN AN ELLIPSOIDAL SHELL OF REVOLUTION

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The stress state around a hole in a shallow shell having positive Gaussian curvature may be described by the solution of the following differential equation (Ref. 3):

$$\nabla^2\nabla^2\Phi^* - ih^{-1}\sqrt{12(1-\nu^2)}\,\nabla_k^2\Phi^* = \frac{Z}{D}. \qquad (1)$$

here $\Phi^* = w^* + i\lambda\phi^*$; w^* is the bending; ϕ^* -- stress function; $\lambda = E^{-1}h^{-2}\sqrt{12(1-\nu^2)}$; D -- cylindrical rigidity; E, ν and h -- elastic constants and shell thickness. The outer loading Z acts in the direction of the outer normal. The operators ∇^2 and ∇_k^2 may be obtained from the operator

$$L^2 = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{M}{A}\cdot\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{N}{B}\cdot\frac{\partial}{\partial\beta}\right)\right],$$

if we set B and A (for the operator ∇^2) or Bk_2 and Ak_1 (for the operator ∇_k^2), instead of M and N. Thus, A and B are the coefficients of the first quadratic form of the shell surface, and α and β are the orthogonal coordinates coinciding with the lines of principal curvatures k_1 and k_2.

Let us investigate a shallow shell, whose middle surface is formed by the revolution of an ellipse with the semiaxes a and b (b -- axis of revolution). We shall assume that Z = const., and that the shell is weakened at the apex by a circular hole having the radius $r = r_0$.

The hole is closed by a cover which transmits only the action of the intersection force to the shell. In addition, we shall only investigate shells which do not differ greatly from spherical shells -- i.e., shells with small eccentricity $\varepsilon = 1 - a^2b^{-2}$.

In view of the shallowness of the shell, it is advantageous to select the polar coordinates r and θ related to the shell apex as α and β. Then A = 1, B = r, and we may take the functions (Ref. 1) as the main radii of curvature /60

$$R_1 = b\left[1 - \varepsilon\left(1 - \frac{3r^2}{2b^2}\right)\right]; \quad R_2 = b\left[1 - \varepsilon\left(1 - \frac{r^2}{2b^2}\right)\right]. \qquad (2)$$

From this point on, we shall discard terms containing powers of ε which are higher than the first power.

If we set $\Phi^* = \tilde{\Phi} + \Phi$, where $\tilde{\Phi}$ is the solution of equation (1) in the case of a shell with no hole, we then have (Ref. 2)

$$\tilde{T}_r = \frac{Zb}{2}\left[1 - \varepsilon\left(1 - \frac{r^2}{2b^2}\right)\right]; \quad \tilde{T}_\theta = \frac{Zb}{2}\left[1 - \varepsilon\left(1 - \frac{3r^2}{2b^2}\right)\right], \qquad (3)$$

The following equation is obtained from (1) for the function Φ expressing the stress state which is perturbed by a hole:

$$\nabla^2\nabla^2\Phi - i\varkappa^2\nabla^2\Phi - i\varkappa^2\varepsilon\frac{1}{r}\cdot\frac{d}{dr}\left(r\frac{d\Phi}{dr} - \frac{r^3}{2b^2}\cdot\frac{d\Phi}{dr}\right) = 0;$$
$$\varkappa = (bh)^{-1}\sqrt{12(1-\nu^2)}. \tag{4}$$

Since only the first powers of ε remain in the expressions for the main radii of curvature, and then in (3), the solution of equation (4) may be naturally sought in the following form

$$\Phi = \Phi^{(0)} + \varepsilon\Phi^{(1)}.$$

In order to find $\Phi^{(0)}$, we then have the equation

$$\nabla^2\nabla^2\Phi^{(0)} - i\varkappa^2\nabla^2\Phi^{(0)} = 0 \tag{5}$$

and the boundary conditions (in stresses and moments)

$$T_r^{(0)} = -\frac{Zb}{2};\quad G_r^{(0)} = 0;\quad Q_r = -\frac{Zr_0}{2} \text{ for } r = r_0. \tag{6}$$

For $\Phi^{(1)}$, the nonhomogeneous equation is obtained

$$\nabla^2\nabla^2\Phi^{(1)} - i\varkappa^2\nabla^2\Phi^{(1)} = i\varkappa^2\frac{1}{r}\cdot\frac{d}{dr}\left(r\frac{d\Phi^{(0)}}{dr} - \frac{r^3}{2b^2}\cdot\frac{d\Phi^{(0)}}{dr}\right) \tag{7}$$

and the boundary conditions

$$T_r^{(1)} = \frac{Zb}{2}\left(1 - \frac{r_0}{2b^2}\right);\quad G_r^{(1)} = 0;\quad Q_r^{(1)} = 0 \text{ for } r = r_0. \tag{8}$$

As may be seen, the problem of the stress state in an ellipsoidal shell of revolution may be reduced to the successive solution of problems regarding the stress state of a spherical shell having the radius b.

The solution of equation (5) is: /61

$$\Phi^{(0)} = (A + iB)H_0^{(1)}(ui\sqrt{i}) + i\lambda\ln u \quad (u = \varkappa r), \tag{9}$$

where $H_0^{(1)}(ui\sqrt{i})$ = her (u) + i hei (u) are the Hankel functions of zero order of the first type. The constants A, B, C for the given boundary conditions (6) have the following form (Ref. 4)

$$A = \frac{Zu_0}{2\varkappa^4 D}\cdot\frac{\beta(u_0)}{\alpha(u_0)\,\mathrm{her}'(u_0 - \beta(u_0)\,\mathrm{hei}'(u_0)};$$
$$B = -\frac{Zu_0}{2\varkappa^4 D}\cdot\frac{\alpha(u_0)}{\alpha(u_0)\,\mathrm{her}'(u_0 - \beta(u_0)\,\mathrm{hei}'(u_0)},\quad C = 0. \tag{10}$$

The functions α and β may be determined by the formulas

$$\alpha(u) = \mathrm{hei}(u) + \frac{1-\nu}{u}\mathrm{her}'(u);\quad \beta(u) = \mathrm{her}(u) - \frac{1-\nu}{u}\mathrm{hei}'(u).$$

The stresses and moments determined according to the function (9) are:

$$T_r^{(0)} = \frac{\varkappa^2 C}{u^2} + \frac{\varkappa}{\lambda}\left(A\frac{\mathrm{hei}'}{u} + B\frac{\mathrm{her}'}{u}\right); \tag{11a}$$

$$T_\theta^{(0)} = -\frac{\varkappa^2 C}{u^2} + \frac{\varkappa^2}{\lambda}\left[A\left(\mathrm{her} - \frac{\mathrm{hei}'}{u}\right) - B\left(\mathrm{hei} + \frac{\mathrm{her}'}{u}\right)\right]; \tag{11b}$$

$$G_r^{(0)} = -D\varkappa^2 [A\alpha(u) + B\beta(u)]; \tag{11c}$$

$$G_\theta^{(0)} = -D\varkappa^2\left[A\left(\nu\,\mathrm{hei} - \frac{1-\nu}{u}\mathrm{her}'\right) - B\left(\nu\,\mathrm{her} + \frac{1-\nu}{u}\mathrm{hei}'\right)\right]. \tag{11d}$$

$$Q_r^{(0)} = D\varkappa^3(A\,\mathrm{hei}' + B\,\mathrm{her}'). \tag{11e}$$

The argument u is omitted here, as well as in the following, in the functions her (u) and hei (u) for the sake of simplicity.

Let us turn to the solution of the problem (7), (8). Equation (7) may be integrated twice, and if we discard the components which we do not need, we obtain

$$\nabla^2\Phi^{(1)} - i\varkappa^2\Phi^{(1)} = i\varkappa^2\left[\Phi^{(0)} - \frac{1}{2b^2}\int r^2\frac{d\Phi^{(0)}}{dr}dr\right],$$

which yields -- after substitution of $\Phi^{(0)}$ from (9) --

$$\nabla^2\Phi^{(1)} - i\varkappa^2\Phi^{(1)} = i\varkappa^2(A+iB)\left[\left(1-\frac{r^2}{2b^2}\right)H_0^{(1)}(\varkappa ri\sqrt{i}) - \right. \\ \left. - \frac{\varkappa r\sqrt{i}}{\varkappa^2 b^2}H_1^{(1)}(\varkappa ri\sqrt{i})\right]. \tag{12}$$

The following function is the general solution of the nonhomogeneous equation (12)

$$\Phi^{(1)} = i\lambda C_1 \ln u + (A_1 + iB_1)H_0^{(1)}(ui\sqrt{i}) - \\ -\frac{1}{2}(A+iB)ui\sqrt{i}H_1^{(1)}(ui\sqrt{i}) + \frac{A+iB}{12\varkappa^2 b^2}[4u^2H_0^{(1)}(ui\sqrt{i}) + \\ + u\sqrt{i}(8+iu^2)H_1^{(1)}(ui\sqrt{i})] \quad (\varkappa r = u). \tag{13}$$

Since it is of particular interest to formulate the particular solution, we may write it separately. If /62

$$\frac{d^2y}{dx^2} + \frac{1}{x}\cdot\frac{dy}{dx} + y = H_0^{(1)}(x);\ x^2H_0^{(1)}(x);\ xH_1^{(1)}(x),$$

then the following functions will be the particular solutions corresponding to these three right hand parts

$$y = \frac{x}{2}H_1^{(1)}(x);\quad \frac{x^2}{b}H_0^{(1)}(x) - \frac{x}{3}\left(1-\frac{x^2}{2}\right)H_1^{(1)}(x); \\ -\frac{x^2}{4}H_0^{(1)}(x) + \frac{x}{2}H_1^{(1)}(x).$$

Dividing the real and imaginary parts in (13), we obtain

$$W^{(1)} = A_1\,\mathrm{her} - B_1\,\mathrm{hei} + \frac{u}{12\varkappa^2 b^2}[4uf(u) + \mu(u)f'(u) - 8\psi'(u)]; \tag{14}$$

$$\lambda\varphi^{(1)} = A_1\,\mathrm{hei} + B_1\,\mathrm{her} + \frac{u}{12\varkappa^2 b^2}[4u\psi(u) + \mu(u)\psi'(u) + 8f'(u)] + \lambda\ln u.$$

Here the following notation is introduced:

$$f(u) = A\,\mathrm{her} - B\,\mathrm{hei}; \quad \psi(u) = A\,\mathrm{hei} + B\,\mathrm{her}; \quad \mu(u) = 6\varkappa^2 b^2 - u^2.$$

The stress and moments corresponding to the stress and bending functions (14) will be:

$$T_r^{(1)} = \frac{\varkappa^2 C_1}{u^2} + \frac{\varkappa^2}{\lambda}\left[A_1 \frac{\mathrm{hei}'}{u} + B_1 \frac{\mathrm{her}'}{u} + t(u)\right]; \tag{15a}$$

$$T_\theta^{(1)} = -\frac{\varkappa^2 C_1}{u^2} + \frac{\varkappa^2}{\lambda}\left[A_1\left(\mathrm{her} - \frac{\mathrm{hei}'}{u}\right) - B_1\left(\mathrm{hei} + \frac{\mathrm{her}'}{u}\right) + T(u)\right]; \tag{15b}$$

$$G_r^{(1)} = -D\varkappa^2 [A_1 \alpha(u) + B_1 \beta(u) + g(u)]; \tag{15c}$$

$$G_\theta^{(1)} = -D\varkappa^2\left[A_1\left(\nu\,\mathrm{hei} - \frac{1-\nu}{u}\mathrm{her}'\right) - B_1\left(\nu\,\mathrm{her} + \frac{1-\nu}{u}\mathrm{hei}'\right) + G(u)\right]; \tag{15d}$$

$$Q_r^{(1)} = D\varkappa^3 [A_1\,\mathrm{hei}' + B_1\,\mathrm{her}' + q(u)]. \tag{15e}$$

The functions thus included t(u); T(u); g(u); G(u); q(u) are given by the following formulas:

$$12\varkappa^2 b^2 t(u) = \mu(u) f(u) + 2u\psi'(u);$$
$$12\varkappa^2 b^2 T(u) = u\mu(u) f'(u) + \mu(u) f(u) + 2u\psi'(u);$$
$$12\varkappa^2 b^2 g(u) = u\mu(u)\psi'(u) + (1+\nu)[\mu(u)\psi(u) - 2uf'(u)];$$
$$12\varkappa^2 b^2 G(u) = \nu u\mu(u)\psi'(u) + (1+\nu)[\mu(u)\psi(u) - 2uf'(u)];$$
$$12\varkappa^2 b^2 q(u) = u\mu(u) f(u) + 2\mu(u)\psi'(u) - 2u^2\psi'(u).$$

The arbitrary constants A_1; B_1; C_1, contained in (14) and (15a) - (15e) may be determined from the boundary conditions (8): /63

$$A_1 = \frac{g(u_0)\,\mathrm{her}' - q(u_0)\,\beta(u_0)}{\alpha(u_0)\,\mathrm{her}' - \beta(u_0)\,\mathrm{hei}'};$$

$$B_1 = \frac{g(u_0)\,\mathrm{hei}' - q(u_0)\,\alpha(u_0)}{\alpha(u_0)\,\mathrm{her}' - \beta(u_0)\,\mathrm{hei}'}; \quad C_1 = 0.$$

By way of an example, two shells were investigated: (1) a = 245 cm, 2) a = 165 cm. The remaining data are identical: b = 200 cm, h = 0.2 cm, r_0 = 20 cm, E = $7.2 \cdot 10^6$ h/cm^2, ν = 0.3. The calculations show that in the first case the stresses on the hole decrease as compared with a spherical shell having the radius b = 200 cm by approximately 12%. In the second case, the stresses T_0^* increase. The moments change to a somewhat greater extent. It may thus be seen that even for small eccentricity its influence may be significant upon the bending stresses.

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3 CONCENTRATION OF STRESSES IN GLASS-FIBER REINFORCED PLASTICS 6 /64

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Synthetic construction materials such as oriented glass-fiber reinforced plastics represent a heterogeneous medium whose main components are elementary glass filaments and a viscous-elastic polymer which connects the filaments to each other. Oriented glass-fiber reinforced plastics which are not made of tissue are of particular practical interest; they make it possible to produce a durable material with sharply expressed anisotropic properties. Due to the nonuniformity of the macrostructure, the stress state of such a material will always be complex due to perturbations produced by rigid inclusions -- glass filaments. Therefore, in glass-fiber reinforced plastics, the stress concentration, arising from openings, grooves, and hollows, applies to perturbation in the stress state in the material itself. All of the polymers presently employed for preparing glass-fiber reinforced plastics are viscous, elastic substances which lead to the redistribution of stresses with time.

A layer of material equipped with a bundle of straightened filaments provides the basis of glass-fiber reinforced plastics which are not made of tissues. The transverse cross section (x 1000) is shown in Figure 1.

In the model which we have assumed, the filaments are placed in an ideal order (Figure 2), forming a regular doubly periodic structure.

At moderate temperatures, the glass filaments represent an elastic material whose mechanical properties may be described by Hooke's law:

$$\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} = 3K_a\varepsilon_{kk},$$
$$\sigma_{ij} - \frac{1}{3}\sigma_{kk}\sigma_{ij} = 2G_a\left(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}\right). \qquad (1)$$

We shall study the thermoreactive polymers used in glass-fiber reinforced plastics on the basis of the theory of elastic hereditary media (Ref. 4). /65

Experiments have shown that the linear theory leads to satisfactory results for a stress up to $0.8\sigma_B$. It was found in experiments that permanent deformations of the polymers examined[1] are quite small. Figure 3 presents the the results derived from an experiment with simple stretching for 98.1·5 bar. The relationship between the stresses and deformations for a homogeneous polymer may be expressed as follows

$$\sigma_{kk} = 3K^*\varepsilon_{kk}; \quad \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} = 2G^*\left(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}\right). \qquad (2)$$

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[1] The experiments were performed on epoxy resins strengthened by maleic anhydride.

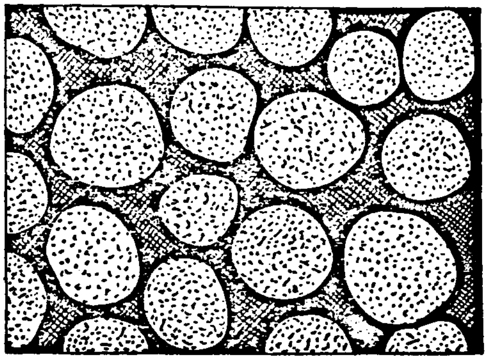

Figure 1

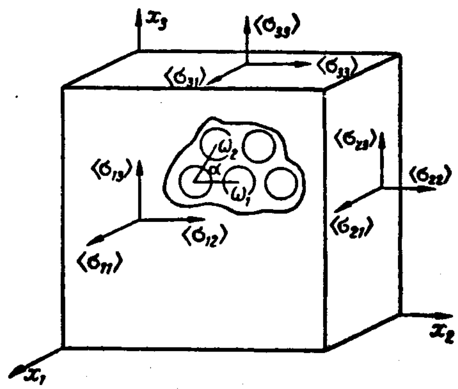

Figure 2

In formulas (2) the quantities K*, G* must be regarded as operators influencing the time function -- for example, /66

$$E^*f(t) = E_0\left\{f(t) - \varkappa_0 \int_0^t f(t')\,\Im_{1-\lambda}(-\varkappa,\ t-t')\,dt'\right\}, \tag{3}$$

where $\Im_{1-\lambda}(-\kappa,\ t-t')$ is the function of Yu. N. Rabotnov (Ref. 4)

$$\Im_{1-\lambda}(-\varkappa,\ t-t') = (t-t')^{\lambda-1}\sum_{k=0}^{\infty}(-\varkappa)^k\frac{(t-t')^{k\lambda}}{\Gamma[\lambda(k+1)]}. \tag{4}$$

Similar formulas exist for other operators.

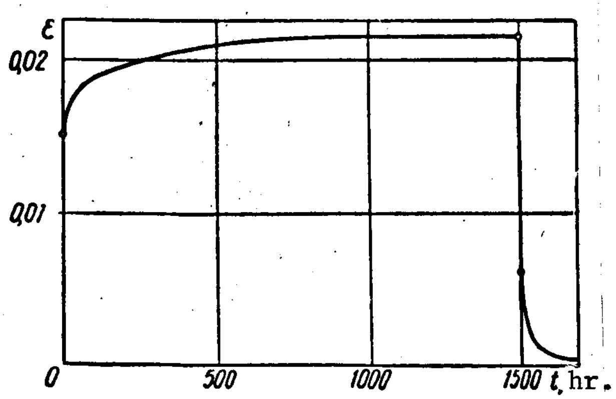

Figure 3

If the stress gradients are such that we may introduce the mean stress on surfaces containing a rather large number of filaments, then the mean stresses and deformations will be related to each other by means of certain relationships. We shall present these relationships here for the case when the resin completely adheres to the filaments and for the case of ideal technology in manufacturing the glass-fiber reinforced plastics.

We shall first establish the relationship between the mean shearing stresses $\langle\tau_{12}\rangle$ and $\langle\tau_{13}\rangle$ and corresponding mean displacement angles.

It may be shown that the solution of the problem regarding the stress state of the reinforced substance shown in Figure 2 may be reduced to determining the two functions (ϕ_a and ϕ_s) of the complex variable $z = x_2 + ix_3$.

The sign $\langle\ \rangle$ designates the mean stress over the averaging area. The /67 functions fulfilling all the requisite conditions of periodicity may be found in the form of the series

$$\varphi_a(z, t) = \sum_{k=0}^{\infty} a_{2k}(t)\frac{z^{2k+1}}{2k+1}, \quad a_{2k} = a'_{2k} + ia''_{2k};$$

$$\varphi_s(z, t) = C_0(t)z - C_2(t)\lambda^2\zeta(z) + \sum_{k=1}^{\infty} C_{2k+2}(t)\frac{\lambda^{2k+2}\wp^{(2k-1)}(z)}{(2k+1)!}; \tag{5}$$

$$C_{2k} = C'_{2k} = iC''_{2k},$$

where $\wp^{(n)}(z)$ is the n^{th} derivative of the doubly periodic (elliptic) Weierstrass function $\wp(z)$; $\zeta(z)$ -- zeta-function of Weierstrass (Ref. 1). The functions ϕ_a and ϕ_s, respectively, determine the stress state of the glass filaments and the resin filling the space between the filaments. The unknowns a_{2k} and C_{2k} may be determined from the coupling conditions at the resin-filament boundary in the case $z = \tau = \lambda e^{i\vartheta}$

$$\varphi_a(\tau, t) + \overline{\varphi_a(\tau, t)} = \varphi_s(\tau, t) + \overline{\varphi_s(\tau, t)}, \quad \varphi_a(\tau, t) - \overline{\varphi_a(\tau, t)} = \frac{G^*}{G}(\varphi_s(\tau, t) - \overline{\varphi_s(\tau, t)}). \tag{6}$$

The real stresses may be determined by means of the mean $\langle\tau_{ik}\rangle$[1]. In particular, the stresses[2] in the glass filaments are

$$\tau_{12} = \frac{2\langle\tau_{12}\rangle}{1+\xi+\eta G^*/G_a} + \frac{2\langle\tau_{12}\rangle}{1+\xi+\eta G^*/G_a}\sum_{k=1}^{\infty} C'_{2k+2}\lambda^{2k+2}\alpha_{0,k} + \frac{2\langle\tau_{12}\rangle}{1-G^*/G_a}\sum_{k=1}^{\infty}\left(\frac{\rho}{\lambda}\right)^{2k} C'_{2k+2}\cos 2k\vartheta + \frac{2\langle\tau_{13}\rangle}{1-G^*/G_a}\sum_{k=1}^{\infty}\left(\frac{\rho}{\lambda}\right)^{2k} C''_{2k+2}\sin 2k\vartheta. \tag{7}$$

Here ξ, η is the volumetric content of the filler and the binder in the composite material; $\frac{\rho}{\lambda} = \frac{r}{r_0}$; r_0 -- filament radius; $\alpha_{i,k}$ -- expansion coefficients of the function $\wp^{(k)}(z)$ in Laurent series. The distribution of stress at the resin-filament boundary in the case of $\frac{\rho}{\lambda} = 1$ is of the greatest interest. It follows from the above formula that

$$\tau_{12} = k_c\langle\tau_{12}\rangle + k'_c\langle\tau_{13}\rangle, \tag{8}$$

where k_c, k'_c are the coefficients of the stress concentration in the material structure, which indicate an increase in the real stress as compared with /68 the average stress. The stresses on other areas may be determined in a

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[1] Since the filament diameter is several micrometers, the linear dimensions of the averaging area must be on the order of tenths of a millimeter.

[2] For considerable stress gradients, we must take into account the moment terms of the expansion of the stress and deformation tensors averaged over the area.

similar manner. If the quantities a_{2k} and C_{2k} are determined, the values of the moduli-operators may be found according to the following formulas

$$X_1^* = 1/G_{12}^* = \frac{\eta + (1+\xi)\, G^*/G_a}{1+\xi+\eta G^*/G_a} \cdot \frac{1}{G^*} - \frac{2\xi\,(1 - G^*/G_a)}{1+\xi+\eta G^*/G_a} \cdot \frac{1}{G^*} \sum_{k=1}^{\infty} C'_{2k+2} \lambda^{2k+2} a_{0,\,k};$$
$$X_3^* = 1/G_{31}^* = \frac{\eta + (1+\xi)\, G^*/G_a}{1+\xi+\eta G^*/G_a} \cdot \frac{1}{G^*} + \frac{2\xi\,(1 - G^*/G_a)}{1+\xi+\eta G^*/G_a} \cdot \frac{1}{G^*} \sum_{k=1}^{\infty} C''_{2k+2} \lambda^{2k+2} a_{0,\,k}. \quad (9)$$

Attention must be called to the fact that the modulus-operators G^*_{12} and G^*_{31}

$G^*_{1^2}$ and $G^*_{3^1}$ will also change significantly with time.

In order to determine the stress state and the moduli under the influence of the stresses $\langle\tau_{23}\rangle$; $\langle\sigma_{11}\rangle$; $\langle\sigma_{22}\rangle$ and $\langle\sigma_{33}\rangle$, we must derive new functions Q(z) (Ref. 9) in order to simplify the solution of the problem:

$$Q(z) = \sum_{m,\,n}{}' \left\{ \frac{\bar{\Pi}}{(z-\Pi)^2} - 2z\frac{\bar{\Pi}}{\Pi^3} - \frac{\bar{\Pi}}{\Pi^2} \right\}; \quad \Pi = m\omega_1 + n\omega_2.$$

The total solution of this problem may be divided into the sum of the solutions of the following problems: plane deformation of a body made of a viscoelastic material with elastic nuclei, and the stress state of a bar when it is loaded by forces in the reinforcing direction.

The desired functions will be found in the form of the expansion

$$\Phi_a(z,\ t) = \sum_{n=0}^{\infty} a_{2n}(t)\, z^{2n}; \quad \Psi_a(z,\ t) = \sum_{n=0}^{\infty} b_{2n}(t)\, z^{2n}, \quad (10)$$

and also for the binder

$$\Phi_s(z,\ t) = C_0(t) + \sum_{k=0}^{\infty} C_{2k+2}(t)\, \frac{\lambda^{2k+2}\, \wp^{(2k)}(z)}{(2k+1)!};$$
$$\Psi_s(z,\ t) = d_0(t) + \sum_{k=0}^{\infty} d_{2k+2}(t)\, \frac{\lambda^{2k+2}\, \wp^{\,k}(z)}{(2k+1)!} - \sum_{k=0}^{\infty} C_{2k+2}(t)\, \frac{\lambda^{2k+2}\, Q^{(2k+1)}(z)}{(2k+1)!}. \quad (11)$$

The unknowns d_{2k}; c_{2k}; a_{2k}; b_{2k} may be determined from the boundary /69
conditions establishing the stress equation at the resin-glass boundary:

$$\Phi_a(\tau,\ t) + \overline{\Phi_a(\tau,\ t)} - [\bar{\tau}\Phi'_a(\tau,\ t) + \Psi_a(\tau,\ t)]\, e^{2i\theta} = \Phi_s(\tau,\ t) + \overline{\Phi_s(\tau,\ t)} - [\bar{\tau}\Phi'_s(\tau,\ t) + \Psi_s(\tau,\ t)]\, e^{2i\theta}. \quad (12)$$

as well as the deformation equation

$$(1-G^*/G_a)\,\Phi_a(\tau,\ t)+(1+\varkappa_a G^*/G_a)\,\overline{\Phi_a(\tau,\ t)}-$$
$$-(1-G^*/G_a)\,[\bar{\tau}\Phi_a'(\tau,\ t)+\Psi_a(\tau,\ t)]\,e^{2i\nu}=(\varkappa^*+1)\,\overline{\Phi_s(\tau,\ t)}. \tag{13}$$

All the real stresses within the material are related to the average stresses by formulas such as (8). The coefficients of the stress concentration in the material structure depend on ξ, η, the mutual location of the filaments, the relative rigidity for the displacement of the resin, the glass filaments, etc.

The explicit values of the modulus-operators may be immediately determined in the case of the specific functions Φ_a; Ψ_a and Φ_s, Ψ_s. In particular, we have

$$X_2^*=1/G_{23}^*=\frac{\varkappa^*\eta+(1+\varkappa^*\xi)\,G^*/G_a}{\xi+\varkappa^*+\eta G^*/G_a}\cdot\frac{1}{G^*}-$$
$$-\frac{\xi(\varkappa^*+1)(1-G^*/G_a)}{\xi+\varkappa^*+\eta G^*/G_a}\cdot\frac{1}{G^*}\left\{\sum_{k=0}^{\infty}C_{2k+2}\lambda^{2k+2}\alpha_{1,\,k}-\right.$$
$$\left.-\sum_{k=0}^{\infty}(2k+2)\,C_{2k+2}\lambda^{2k+2}\beta_{0,\,k}+\sum_{k=0}^{\infty}d_{2k+2}\lambda^{2k+2}\alpha_{0,\,k}\right\}. \tag{14}$$

The general form of the relationship between the stresses and the mean deformations for a three-dimensional stress state is

$$\langle\varepsilon_{12}\rangle=X_1^*\langle\tau_{12}\rangle;\quad\langle\varepsilon_{23}\rangle=X_2^*\langle\tau_{23}\rangle;\quad\langle\varepsilon_{31}\rangle=X_3^*\langle\tau_{31}\rangle; \tag{15a}$$
$$\langle\varepsilon_{11}\rangle=X_{11}^*\langle\sigma_{11}\rangle+X_{12}^*\langle\sigma_{22}\rangle+X_{13}^*\langle\sigma_{33}\rangle; \tag{15b}$$
$$\langle\varepsilon_{22}\rangle=X_{21}^*\langle\sigma_{11}\rangle+X_{22}^*\langle\sigma_{22}\rangle+X_{23}^*\langle\sigma_{33}\rangle; \tag{15c}$$
$$\langle\varepsilon_{33}\rangle=X_{31}^*\langle\sigma_{11}\rangle+X_{32}^*\langle\sigma_{22}\rangle+X_{33}^*\langle\sigma_{33}\rangle. \tag{15d}$$

The values of X_1^*, X_2^* and X_3^* may be determined by formulas (9) and (14). The remaining modulus-operators may be found in the following form

$$(X_{11}^*)^{-1}=\xi E_a+\eta E^*+$$
$$+8\xi G^*(\nu_a-\nu^*)^2\,\frac{\eta+(\varkappa^*+1)\sum_{k=1}^{\infty}C_{2k+2}\lambda^{2k+2}\alpha_{0,\,k}}{1+\eta+\xi\varkappa^*+\eta(\varkappa_a-1)\,G^*/G_a}; \tag{16a}$$
$$X_{21}^*=X_{11}^*\{-\nu^*+(\varkappa^*+1)(\nu_a-\nu^*)(C_0-\xi C_2)\}; \tag{16b}$$
$$X_{31}^*=X_{11}^*\{-\nu^*+(\varkappa^*+1)(\nu_a-\nu^*)(C_0+\xi C_2)\}; \tag{16c}$$ /70
$$X_{22}^*=\frac{1}{8G^*}(\varkappa^*+1)(C_0-\xi C_2'+1)+$$
$$+X_{11}^*Y^*\{\nu^*-(\nu_a-\nu^*)(\varkappa^*+1)(C_0'-\xi C_2')\}; \tag{16d}$$
$$X_{32}^*=\frac{1}{8G^*}\{(\nu^*+1)(C_0+\xi C_2)+\varkappa^*-3\}+$$
$$+X_{11}^*Y^*\{\nu^*-(\nu_a-\nu^*)(\varkappa^*+1)(C_0'+\xi C_2')\}; \tag{16e}$$
$$X_{33}^*=\frac{1}{8G^*}(\varkappa^*+1)(C_0+\xi C_2+1)+$$

$$+ X_{11}^{*}Y^{*}\{\nu^{*} - (\nu_a - \nu^{*})(\varkappa^{*} + 1)(C_0' + \xi C_2')\}; \qquad (16f)$$

$$Y^{*} = \nu^{*} + (\nu_a - \nu^{*})(\xi - \eta C_0 + \xi \sum_{k=0}^{\infty} C_{2k+2}\lambda^{2k+2}a_{0,k}). \qquad (16g)$$

These formulas clearly illustrate the dependence of the modulus-operator on the inner structure of the material and the viscoelastic properties of the binders and the mechanical characteristics of the fillers. The form of relationships (15a) - (15d) coincides with the equations of Hooke's law for an orthotropic body, which may be explained by the structural symmetry of the material with respect to the coordinate axes. Formulas (16a) -(16g) must be simplified in order to compile equations for the theory of shells made of glass-fiber reinforced plastics. Experimental investigations on a homogeneous polymer have shown that the Poisson coefficient $0.385 < \nu < 0.42$ -- i.e., it may be assumed that the operators X_{21}^{*}, X_{31}^{*} change very little with time for reinforced plastics. The portion of loading which may be received by the resin on the surface x_1 = const. is very small. With sufficient approximation, we may therefore assume that the modulus operators X_{ik}^{*}, with the exception of X_{22}^{*} and X_{33}^{*}, do not change with time. The instantaneous (elastic) values of the moduli were compared with experimental data. The greatest deviation was observed for X_{11}^{*}, which may be explained by the influence of uneven tension of the glass filaments; the remaining moduli values satisfactorily agree with the experimental results.

Shells made of non-tissue, glass-fiber reinforced plastics were prepared by the method of consecutive super-position of layers reinforced in the direction chosen. Materials with glass-filaments which were oriented perpendicular to each other received the greatest propagation (Figure 4). For the case of moderate intersection stresses, thin shells were investigated on the basis of the theory of laminar shells (Ref. 2), in which the Kirchhoff-Love hypothesis is applied for the entire packet as a whole. If a uniform stress state is given in the shell, a local stress state with a large variability coefficient arises around the hole (Ref. 7, 8). In order to study the stress concentration around a hole, let us introduce a local coordinate system α, β, which may be regarded as a plane system for a sufficiently small hole. The basic system of differential equations for studying the stress concentration around holes in shells has the following form (Ref. 2) /71

$$\begin{aligned} L_1(D_{ij}^{*} - D_{ij}^{0*})w - L_3(d_{ij}^{*})\varphi + \nabla_k\varphi = q; \\ L_2(A_{ij}^{*})\varphi + L_4(d_{ij}^{*})w - \nabla_k w = 0, \end{aligned} \qquad (17)$$

where w and ϕ are the functions of bending and stress. The mechanical characteristics (16a) - (16g) are employed to determine the operators. It may be seen, in the general case the system of equations (17) for a plate (in the case $\nabla_k = 0$) does not decompose into two independent equations. For material which is composed of layers symmetrical to the middle surface, the main system of equations assumes the form

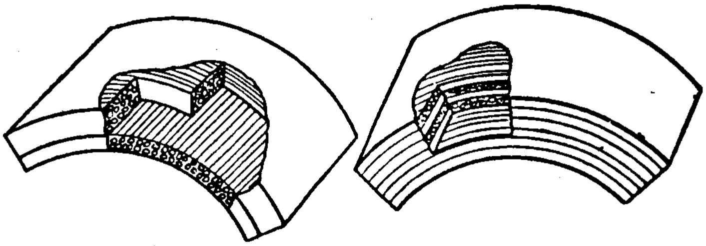

Figure 4

$$L_1(D_{ij}^*)\,w + \nabla_k\varphi = q;\qquad L_2(A_{ij}^*)\,\varphi - \nabla_k w = 0. \tag{18}$$

For rather small holes, when we may disregard the influence of the middle surface curvature, the stress concentration around the hole may be determined from the solution of the following equation, in the case of uniform membrane stresses in the plate

$$\frac{\partial}{\partial\alpha}\cdot\frac{1}{A}\left\{B\frac{\partial}{\partial\alpha}J(A_{22}^*) + \frac{\partial B}{\partial\alpha}[J(A_{22}^*) - J(A_{11}^*)] - \frac{A}{2}\cdot\frac{\partial}{\partial\beta}J(A_{66}^*) - \right.$$
$$\left. - \frac{\partial A}{\partial\beta}J(A_{66}^*)\right\}\varphi + \frac{\partial}{\partial\beta}\cdot\frac{1}{B}\left\{A\frac{\partial}{\partial\beta}J(A_{11}^*) + \frac{\partial A}{\partial\beta}[J(A_{11}^*) - J(A_{22}^*)] - \right.$$
$$\left. - \frac{B}{2}\cdot\frac{\partial}{\partial\alpha}J(A_{66}^*) - \frac{\partial B}{\partial\alpha}J(A_{66}^*)\right\}\varphi = 0,$$

where the following notation is employed (Ref. 2)

$$J(A_{11}^*) = \frac{C_{22}^*}{\Omega^* B}\left[\frac{\partial}{\partial\beta}\cdot\frac{1}{B}\cdot\frac{\partial}{\partial\beta} + \frac{1}{A^2}\cdot\frac{\partial B}{\partial\alpha}\cdot\frac{\partial}{\partial\alpha}\right] -$$
$$- \frac{C_{12}^*}{\Omega^* A}\left[\frac{\partial}{\partial\alpha}\cdot\frac{1}{A}\cdot\frac{\partial}{\partial\alpha} + \frac{1}{B^2}\cdot\frac{\partial A}{\partial\beta}\cdot\frac{\partial}{\partial\beta}\right];\quad J(A_{22}^*) = \frac{C_{11}^*}{\Omega^* A}\left[\frac{\partial}{\partial\alpha}\cdot\frac{1}{A}\cdot\frac{\partial}{\partial\alpha} + \right.$$
$$\left. + \frac{1}{B^2}\cdot\frac{\partial A}{\partial\beta}\cdot\frac{\partial}{\partial\beta}\right] - \frac{C_{12}^*}{\Omega^* B}\left[\frac{\partial}{\partial\beta}\cdot\frac{1}{B}\cdot\frac{\partial}{\partial\beta} + \frac{1}{A^2}\cdot\frac{\partial B}{\partial\alpha}\cdot\frac{\partial}{\partial\alpha}\right];$$
$$J(A_{66}^*) = -\frac{1}{C_{66}^* AB}\left[\frac{\partial^2}{\partial\alpha\,\partial\beta} - \frac{1}{A}\cdot\frac{\partial A}{\partial\beta}\cdot\frac{\partial}{\partial\alpha} - \frac{1}{B}\cdot\frac{\partial B}{\partial\alpha}\cdot\frac{\partial}{\partial\beta}\right].$$

By way of an example, let us study the stress concentration around a circular hole in a plate which is reinforced in one direction in the case of uniaxial tension. For the given case, equation (19) in Cartesian coordinates α, β assumes the following form /72

$$X_{22}^*\frac{\partial^4\varphi}{\partial\alpha^4} + (X_1^* + 2X_{21})\frac{\partial^4\varphi}{\partial\alpha^2\,\partial\beta^2} + X_{11}\frac{\partial^4\varphi}{\partial\beta^4} = 0. \tag{20}$$

If we set

$$X_{11} = 1/E_1;\quad X_{22}^* = 1/E_2^*;\quad X_1^* = 1/G^*;\quad X_{12}^*/X_{11} = -\nu_1^*,$$

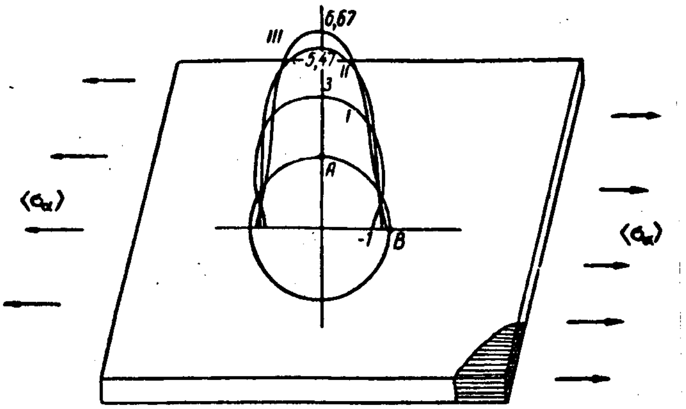

Figure 5

then the form of (20) coincides with the resolvent in the theory of orthotropic plates (Ref. 3, 6), and the integral operators play the role of the coefficients. A solution of this problem in the elastic region was obtained in (Ref. 3, 6) based on the theory of the function of the complex variable $z_1 = \alpha + S_1\beta$ and $z_2 = \alpha + S_2\beta$, where

$$s_1 s_2 = -\sqrt{E_1/E_2}; \quad s_1 + s_2 = i\sqrt{2(\sqrt{E_1/E_2} - \nu_1) + E_1/G}.$$

In order to study the stress concentration around a hole in a viscous elastic body, the mechanical characteristics must be replaced by the integral operators in the final formulas obtained for the elastic problem. The solution of the integral equations compiled makes it possible to determine the change in stress with time.

The normal stresses $\langle\sigma_\nu\rangle$ acting upon radial regions are as follows at the hole profile:

$$\langle\sigma_\nu\rangle = X_\vartheta^* \left\{\sqrt{X_{11}X_{22}^*}\cos^2\vartheta + \right.$$
$$\left. + \left(X_{11} + \sqrt{2X_{11}(X_{22}^* + X_{21}) + X_1^* X_{11}}\right)\sin^2\vartheta\right\}, \langle\sigma_\alpha\rangle, \tag{21}$$

where we set /73

$$X_\vartheta^* = X_{11}\sin^4\vartheta + (X_1^* + 2X_{21})\sin^2\vartheta\cos^2\vartheta + X_{22}^*\cos^4\vartheta.$$

If the uniform stress state does not change with time, then -- by employing the known approximations for the function $Э_{1-\lambda}(-\kappa, t)$ (Ref. 5) -- we may determine the stress redistribution in time directly from (21).

On the basis of experiments, we found $E_a = 0.981 \cdot 7 \cdot 10^5$ bar; $\nu_a = 0.2$, $\lambda = 0.5$; $\nu_0 \approx 0.4$; $\kappa_0 = 0.057\ h^{-\lambda}$; $\kappa = 0.177\ h^{-\lambda}$; $E_0 = 0.981 \cdot 3 \cdot 10^4$ bar.

Figure 5 presents the results derived from a numerical calculation of the stress $\langle\sigma_{\vartheta}\rangle$ along the hole profile (in the case $\xi = 0.74$). Curve I is given for isotropic material, II -- for glass-fiber reinforced plastic in the elastic region, and III for glass-fiber reinforced plastic 500 hours at the loading σ_{α} = const. The real stress in the material may be represented in the following form

$$\sigma_{\vartheta} = k_c \langle\sigma_{\vartheta}\rangle = k_c k_0 \langle\sigma_{\alpha}\rangle.$$

For glass-fiber reinforced plastics, k_c depends slightly on time, and k_0 changes significantly with time, as may be seen from Figure 5.

An investigation has shown that at point A (see Figure 5) the stress σ_{ϑ} is primarily absorbed by the glass filaments. In this region, $k_c \approx 1$ for the filament, at the point B at the resin-glass boundary the largest value $k_c = 1.47$ occurs.

Thus, the stress concentration around a hole in synthetic materials such as glass-fiber reinforced plastics may be determined by the curvature of the hole profile and by the surface curvature (for shells), by the material anistropy, by the viscoelastic properties of the glass-fiber reinforced plastics, and by the structural coefficient of the stress concentration k_c characterizing the stress distribution in the material between the filler and the binder.

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ELASTOPLASTIC STATE NEAR A REINFORCED HOLE IN A SPHERICAL SHELL /74

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Let us investigate the stress state and state of deformation in the zone of stress concentration around a circular hole in a spherical shell subjected to internal pressure (Figure 1). The spherical shell is changed into a circular, toroidal shell close to the hole, and is reinforced by a narrow elastic ring. Only vertical stresses are transmitted from the lid covering the hole to the ring.

The solution of the problem beyond the elasticity limit is based on the theory of small elastoplastic deformations, the method of elastic solutions (Ref. 3), and the method of finite differences.

The axisymmetric, elastoplastic state of the shell of revolution may be described by the following differential equations (Ref. 1):

$$
\begin{gathered}
m_{10}u + m_{11}u' + m_{12}u'' + n_{10}w + n_{11}w' + n_{12}w'' - w''' + \\
+ A^4D^{-1}X + \Omega_u = 0; \\
m_{20}u + m_{21}u' + m_{22}u'' + n_{20}w + n_{21}w' + n_{22}w'' + n_{23}w''' - w^{IV} + \\
+ A^4D^{-1}Z + \Omega_w = 0.
\end{gathered}
$$

For a constant meridian curvature and incompressibility of the shell material, the coefficients of the equations may be determined by the following formulas

$$
\begin{gathered}
m_{10} = 6\eta^2(\bar{B}'' - 2\bar{B}'^2) + 0{,}5(\bar{k}_2'' - \bar{B}'\bar{k}_2'); \quad m_{11} = 12\eta^2\bar{B}' + 0{,}5\bar{k}_2', \\
m_{12} = 12\eta^2; \quad n_{10} = 12\eta^2\bar{B}'(1 - \bar{k}_2) + 0{,}5\bar{B}'(\bar{k}_2^2 - 1) - \bar{k}_2\bar{k}_2'; \\
n_{11} = 12\eta^2(1 + 0{,}5\bar{k}_2) - 0{,}5(\bar{B}'' + \bar{k}_2^2) + \bar{B}'^2 - 1; \quad n_{12} = -\bar{B}'; \\
m_{20} = -6\eta^2\bar{B}'(1 + 2\bar{k}_2) - 0{,}5(\bar{B}''\bar{k}_2' - \bar{k}_2''');
\end{gathered}
$$

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$$
\begin{gathered}
m_{21} = -6\eta^2(2 + \bar{k}_2) + \bar{k}_2; \quad m_{22} = 0{,}5\bar{k}_2'; \\
n_{20} = -12\eta^2(1 + \bar{k}_2 + \bar{k}_2^2) + 0{,}5\bar{B}''(\bar{k}_2^2 - 1) - \bar{k}'^2_2 - \bar{k}_2\bar{k}_2''; \\
n_{21} = \bar{B}'(2\bar{B}'' - \bar{B}'^2 - 1) - 2\bar{k}_2\bar{k}_2'; \quad n_{22} = \bar{B}'^2 - 0{,}5(3\bar{B}'' + \bar{k}_2^2 + 2); \\
n_{23} = -2\bar{B}',
\end{gathered}
$$

where

$$
\eta = \frac{A}{h}, \ \bar{B}' = \frac{B'}{B}, \ \bar{B}'' = \frac{B''}{B}, \ \bar{k}_2 = Ak_2, \ \bar{k}_2' = Ak_2'; \ \bar{k}_2'' = Ak_2''.
$$

The following notation is also employed in these equations: u, w, X, Z -- components of displacement and loading intensity, respecitvely, in the direction which is tangent and normal to the shell meridian; D -- cylindrical rigidity; A, B -- coefficients of the first quadratic form; $\bar{k}_2$ -- principal curvature in the peripheral direction; h -- shell thickness.

For the spherical shell, we have (see Figure 1): $A = R_c$, $B = R_c \sin \phi$, $k_2 = R_c^{-1}$, and for the toroidal shell we have $A = R_T$,

$$B = (t + \sin \varphi) R_T, \quad k_2 = B^{-1} \sin \varphi, \quad t = aR_T^{-1}.$$

The expressions for the non-linear portion of the equations have the following form

$$\begin{aligned}
\Omega_u &= A^3 D^{-1} [\bar{B}' (\Delta T_1 - \Delta T_2) + \Delta T_1' + \Delta Q_1];\\
\Omega_w &= A^3 D^{-1} [-(\Delta T_1 + \bar{k}_2 \Delta T_2) + \bar{B}' \Delta Q_1 + \Delta Q_1'];\\
\Delta T_1 &= -12 D \eta A^{-2} [\eta J_1 (\varepsilon_1 + 0{,}5\varepsilon_2) + A J_2 (\varkappa_1 + 0{,}5\varkappa_2)];\\
\Delta T_2 &= -12 D \eta A^{-2} [\eta J_1 (\varepsilon_2 + 0{,}5\varepsilon_1) + A J_2 (\varkappa_2 + 0{,}5\varkappa_1)];\\
\Delta G_1 &= 12 D A^{-1} [\eta J_2 (\varepsilon_1 + 0{,}5\varepsilon_2) + A J_3 (\varkappa_1 + 0{,}5\varkappa_2)];\\
\Delta G_2 &= 12 D A^{-1} [\eta J_2 (\varepsilon_2 + 0{,}5\varepsilon_1) + A J_3 (\varkappa_2 + 0{,}5\varkappa_1)];\\
\Delta Q_1 &= A^{-1} [\bar{B}' (\Delta G_2 - \Delta G_1) - \Delta G_1'];\\
J_1 &= \int_{-0,5}^{0,5} \omega d\xi; \quad J_2 = \int_{-0,5}^{0,5} \omega \xi d\xi; \quad J_3 = \int_{-0,5}^{0,5} \omega \xi^2 d\xi; \quad \xi = \frac{z}{h}.
\end{aligned}$$

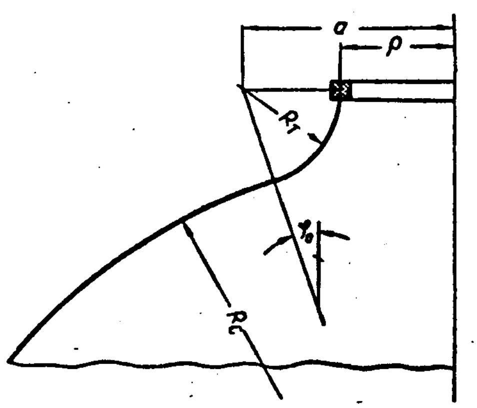

Figure 1

Here ε_1, ε_2, χ_1, χ_2 represent the deformation of the middle surface (Ref. 2); ω -- plasticity function of Il'yushin (Ref. 3); ξ -- relative coordinate with respect to shell thickness.

Two compatibility equations of the ring and shell displacements and the equation of the ring equilibrium are the boundary conditions for the reinforced edge of the torus. The compatibility conditions for displacement and equality of the internal stresses in the associated cross-section of the sphere and the torus yield six equations. A membrane state is assumed in the spherical shell far from the perturbation zone. The meridian and circular stresses may be calculated according to the following formulas /76

$$\sigma_1 = \frac{4E(1-\omega)}{3} (e_1 + 0{,}5e_2);$$

$$\sigma_2 = \frac{4E(1-\omega)}{3} (e_2 + 0{,}5e_1),$$

where

$$e_1 = \varepsilon_1 + \varkappa_1 z; \quad e_2 = \varepsilon_2 + \varkappa_2 z.$$

The elastoplastic state of the system was studied in the case $R_c : h = 400$, $R_T : h = 13(3)$, $R_T : R_c = 0.03(3)$, $\rho : R_T = 6.5$, $t = 7.5$, $\phi_0 = 0.244360$, $F_K : \rho^2 = 0.0133136$, $J_K : \rho^4 = 0.169593 \cdot 10^{-4}$, where F_K, J_K represent the area of the ring cross section and the principal moment of inertia with respect to the horizontal, central axis. The material of the shell ring is steel Sp 3 ($\sigma_{r*} = 200$ Mn/m^2, $\varepsilon_r = 0.001$, $\sigma_T = 240$ Mn/m^2, $\varepsilon_T = 0.006$). The curve given in the figure has a smooth transition in the deformation range 0.001 - 0.006.

The finite difference equations of equilibrium, of boundary conditions, and the associated conditions were compiled for the step of the independent variable $\lambda_s = 0.436332 \cdot 10^{-2}$ rad for the sphere, and $\lambda_T = 0.0736905$ rad for the torus. The step magnitude was selected by comparing the solutions for different steps obtained for a spherical shell with a reinforced hole. The step for the torus was taken so that the lengths of the arcs of the sphere and the torus, corresponding to the assumed steps, were approximately the same.

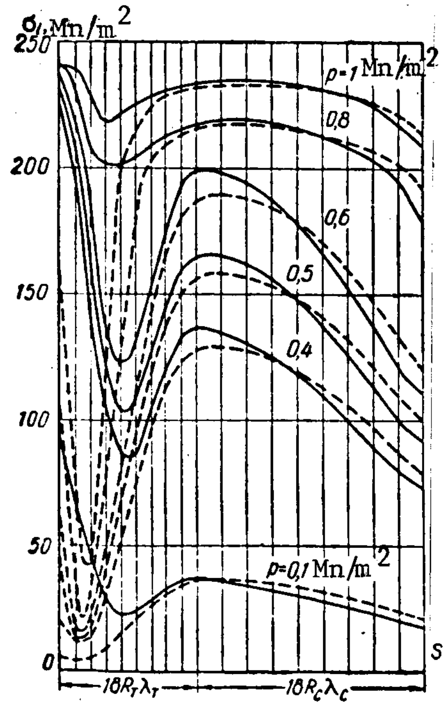

Figure 2

The membrane boundary conditions for the spherical portion of the system were assumed at a distance of $21R_c\lambda_c$ from the associated cross-section with the torus.

The numerical solution of the problem was obtained on a high-speed electronic 2M computer. The program compiled included the calculation of the coefficients of a system of finite difference equations, the right hand parts, the nonlinear part of the equations, and also it included subroutines for individual parts of the calculation when the problem was solved by successive approximations. A standard program was developed in the Institute of Mechanics of the USSR Academy of Sciences by M. I. Dlugach and A. S. Stepanenko for solving an asymmetrical system of linear algebraic equations having a band structure. /77

The numerical results of the solution are presented in the form of curves showing the change in the stress intensity σ_i for an internal pressure of 0.1 to 1 Mn/m^2 (Figure 2). The arc of the system meridian, beginning with the ring, is plotted on the abcissa axis. One division within the torus ($18R_T\lambda_T$) corresponds to an arc equalling $2R_T\lambda_T$, and within the sphere $2R_c\lambda_c$.

*Translator's note: This probably designates "reinforced".

The solid line shows the change in σ_i on the inner surface of the system, and the dashed line shows the change on the middle surface. The curve σ_i on the outer surface almost coincides with the curve on the inner surface, and therefore is not shown in Figure 2.

It may be seen from the graph that σ_i on the middle surface of the torus in the region adjoining the ring is considerably lower than on the outer surfaces of the shell. Thus, for a pressure which produces membrane stresses close to σ_T in the sphere, σ_i is less than σ_r.

The table presents the ratio of the smallest stress intensity to the membrane stress for the toroidal (k_T) and spherical (k_s) parts of the system and the largest deformation intensity existing in the system for certain pressures.

(1)	p·Mn/m^2								
	0,1	0,4	0,5	0,6	0,7	0,8	0,9	1,0	1,1
k_T	4,66	2,82	2,33	1,98	1,71	1,50	1,33	1,20	1,09
k_c	1,82	1,70	1,68	1,67	1,49	1,39	1,28	1,17	1,08
$e_i \cdot 10^2$	0,047	0,21	0,31	0,41	0,56	0,68	0,89	1,11	1,34

(1) - Characteristics

The greatest stresses arise in the associated cross section of the torus and the ring. The stress concentration in a spherical shell is somewhat smaller. The distance between the concentration coefficients for the torus and the sphere decreases with an increase in pressure, and their magnitudes sharply decrease.

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M 3

3 SLIP BANDS IN THIN PLATES WITH RECTILINEAR CUTS UNDER TENSION /78

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N67-24509

Plate with two equal slits. Let a thin, infinite plate with two equal rectilinear cuts (narrow slits) along the segments (-b, -a) and (a, b) of the abscissa axis be pulled at infinity by the stresses $\sigma_y^\infty = p$ (Figure 1). We shall assume that the plate material is isotropic, elastic-ideally plastic with a flow limit under simple tension σ_T. The condition of constant, maximum, shearing stresses

$$\tau_{max} = \frac{1}{2}\sigma_T \tag{1}$$

is assumed as the plasticity condition.

As is known, due to the fact that there no limitations on the stresses, at the ends of the cuts the elasticity conditions cannot be satisfied for any small loading (except for p > 0). Therefore, plastic deformations occur here.

Experiments have shown (Ref. 2, 5) that plastic deformations during the first stages of their development are localized in narrow slip bands occupying an insignificant area as compared with the elastic portion of the body, when there is a sufficiently nonhomogeneous stress field. This development of plastic deformations is particularly characteristic for materials having a well defined flow region.

In the case under consideration, we shall study the development of the first slip bands formed in the vicinity of the cut ends.

We shall employ the following method (Ref. 1, 6) in order to study the development of slip bands analytically. In view of the small width of these bands, we shall assume that the plastic deformation is concentrated along certain lines. We must thus assume a displacement discontinuity on these lines. The assumed displacement discontinuities must not contradict the plastic flow mechanism which is kinematically possible and need not imply the presence of cracks (cavities) in the body, if the cracks are formed as a result of plastic deformation -- i.e., if the material remains compact. In thin plates, discontinuities which are both tangential and normal to the discontinuity lines can satisfy these requirements. In the case of a normal discontinuity, the disturbance of the compact nature of the material may be due exclusively to local modification or thickening of the plate. /79

Thus, the local nature of the development of slip bands means that the problem of the elastic-plastic equilibrium of the plate may be reduced to the problem of the equilibrium of an elastic plate whose displacements undergo a discontinuity along specific lines. The stresses acting along these lines must satisfy the plasticity condition. The form and length of the discontinuity lines (slip lines) are not known ahead of time, and must be determined as the problem is being solved. In several special cases, the form of these lines can

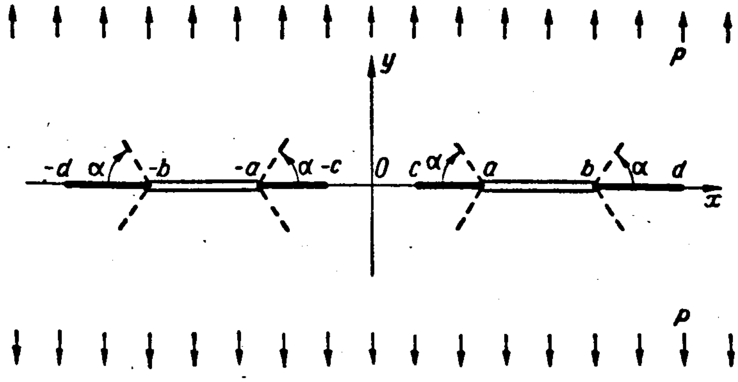

Figure 1

be predicted. The solution of the problem is then significantly facilitated.

In our case, the first slip bands are propagated along the extension lines of the slits (along the line $y = 0$, $|x| \leq a$, $|x| \geq b$), where, as is known from the elastic solution of the problem, the stresses are maximum, and we have

$$\sigma_y(x, 0) > \sigma_x(x, 0) > \sigma_z(x, y) = 0; \quad \tau_{xy}(x, 0) = 0. \tag{2}$$

In view of these relationships, the plasticity condition (1) on the abscissa axis assumes the following form

$$\sigma_y(x, 0) = \sigma_T. \tag{3}$$

The slip regions coincide with the Ox-axis, and are inclined at the angle $\pm 45°$ to the plate plane. As a result of the shifts, local modification of the plate arises in certain segments of the abscissa axis $c \leq |x| \leq a$ and $b \leq |x| \leq d$, which are occupied by the first slip bands. /80

According to the statements given above, we may reduce the problem of the development of these bands to the problem of the elasticity theory with discontinuous displacements $v(x, 0)$ on the segments $c \leq |x| \leq a$ and $b \leq |x| \leq d$, in which condition (3) is satisfied. Let us deal with this problem, continuing the slits in these segments and applying the stresses $\sigma_y(x, \pm 0) = \sigma_T$ to the edges of this continuation -- i.e., investigating the extension of the elastic plate with cuts along the segments $c \leq |x| \leq d$ under the following boundary conditions:

$$\tau_{xy} = (t, \pm 0) = 0 \text{ for } t \text{ on } L, \quad \sigma_y(t, \pm 0) = p(t) = \begin{cases} 0 & \text{for } t \text{ on } L' \\ \sigma_T & \text{for } t \text{ on } L'' \end{cases} \tag{4}$$

where

$$L = (-d, -c) + (c, d), \quad L' = (-b, -a) + (a, b), \quad L'' = L - L'.$$

The ends of the slip bands -- i.e., the points $|x| = c$ and $|x| = d$ -- must be determined so that the condition of the boundedness and the continuity of stresses is fulfilled.

In the book by N. I. Muskhelishvili (Ref. 4), the general solution of the first main problem for a plane with rectilinear cuts is given by two complex functions $\Phi(z)$ and $\Omega(z)$, which have the following form in our case

$$\Phi(z) = \Phi_0(z) + \frac{P_2(z)}{X(z)} - \frac{1}{4}p, \quad \Omega(z) = \Omega_0(z) + \frac{P_2(z)}{X(z)} + \frac{1}{4}p \quad (z = x + iy), \tag{5}$$

where

$$\Phi_0(z) = \Omega_0(z) = \frac{1}{2\pi i X(z)} \int_L \frac{X^+(t)\, p(t)}{t - z} dt = \frac{\sigma_T}{2\pi i X(z)} \int_{L''} \frac{X^+(t)}{t - z} dt; \tag{6}$$

$$P_2(z) = c_0 z^2 + c_1 z + c_2, \quad c_0 = \frac{1}{2}p; \quad X(z) = \sqrt{(z^2 - c^2)(z^2 - d^2)}. \tag{7}$$

The quantity $X(z)$ is used to designate the branch which is holomorphic in the plane with the cuts, so that $X(z) \to z^2$ in the case $|z| \to \infty$.

The coefficients c_1 and c_2 may be determined from the equation

$$2\int_{L_k} \frac{P_2(t)}{X^+(t)} dt + \int_{L_k} [\Phi_0^+(t) - \Phi_0^-(t)]\, dt = 0 \quad (k = 1, 2), \tag{8}$$

where $L_1 = (-d, -c)$, $L_2 = (c, d)$, and the indices + and - designate the boundary values of the quantities when approaching L_k from the left and the right, respectively.

Calculating the integrals in formulas (6) and (8), we obtain /81

$$\begin{aligned}\Phi_0(z) = \Omega_0(z) = {} & \frac{\sigma_T}{2\pi \sqrt{(z^2 - c^2)(z^2 - d^2)}} \Big\{ \sqrt{(a^2 - c^2)(d^2 - a^2)} - \\ & - \sqrt{(b^2 - c^2)(d^2 - b^2)} - \frac{d^2 + c^2 - 2z^2}{2} \Big(-\pi + \arcsin \frac{-2a^2 + d^2 + c^2}{d^2 - c^2} - \\ & - \arcsin \frac{-2b^2 + d^2 + c^2}{d^2 - c^2} \Big) - \\ & - 2\sqrt{(z^2 - c^2)(d^2 - z^2)} \ln \frac{[\sqrt{(z^2 - c^2)(d^2 - a^2)} + \sqrt{(d^2 - z^2)(a^2 - c^2)}]\sqrt{b^2 - z^2}}{[\sqrt{(z^2 - c^2)(d^2 - b^2)} + \sqrt{(d^2 - z^2)(b^2 - c^2)}]\sqrt{z^2 - a^2}} \Big\};\end{aligned} \tag{9}$$

$$\begin{aligned} & c_1 = 0; \\ & c_2 = \frac{\sigma_T}{2\pi} \Big[\sqrt{(d^2 - a^2)(a^2 - c^2)} - \sqrt{(d^2 - b^2)(b^2 - c^2)} + \\ & + \frac{d^2 + c^2}{2} \Big(-\pi + \arcsin \frac{-2a^2 + d^2 + c^2}{d^2 - c^2} - \arcsin \frac{-2b^2 + d^2 + c^2}{d^2 - c^2} \Big) \Big] - \end{aligned} \tag{10}$$

$$
\begin{gathered}
-\frac{d}{F(k)}\left\{dE(k)\left[c_0+\frac{\sigma_T}{2\pi}\left(-\pi+\arcsin\frac{-2a^2+d^2+c^2}{d^2-c^2}-\right.\right.\right. \\
\left.\left.-\arcsin\frac{-2b^2+d^2+c^2}{d^2-c^2}\right)\right]+\frac{\sigma_T}{\pi}\sqrt{(d^2-b^2)(b^2-c^2)}\times \\
\times\frac{b^2}{d(d^2-b^2)}\prod\left(-\frac{d^2-c^2}{d^2-b^2},k\right)-\frac{\sigma_T}{\pi}\sqrt{(d^2-a^2)(a^2-c^2)}\times \\
\left.\times\frac{a^2}{d(d^2-a^2)}\prod\left(-\frac{d^2-c^2}{d^2-a^2},k\right)\right\},
\end{gathered} \tag{10}
$$

where F, E, Π designate the complete elliptic integrals of the first, second, and third types, respectively; $k = \frac{\sqrt{d^2 - c^2}}{d}$.

The stress field may be determined from the following formula (Ref. 4)

$$
\begin{gathered}
\sigma_y+\sigma_x=2[\Phi(z)+\overline{\Phi(z)}]; \\
\sigma_y-\sigma_x+2i\tau_{xy}=2[\Omega(\bar{z})-\Phi(z)-(z-\bar{z})\Phi'(z)]\quad(\bar{z}=x-iy).
\end{gathered} \tag{11}
$$

Based on formulas (5), (7), (9), (11), we may readily show that we must set the following in order to insure the boundedness and continuity of the stress at the points $z = \pm c$ and $z = \pm d$:

$$
\begin{gathered}
\frac{\sigma_T}{2\pi}\left(-\pi+\arcsin\frac{-2a^2+d^2+c^2}{d^2-c^2}-\arcsin\frac{-2b^2+d^2+c^2}{d^2-c^2}\right)+c_0=0; \\
\frac{\sigma_T}{2\pi}[\sqrt{(a^2-c^2)(d^2-a^2)}-\sqrt{(b^2-c^2)(d^2-b^2)}]+\frac{d^2+c^2}{2}c_0+c_2=0.
\end{gathered} \tag{12}
$$

Substituting the values $c_0 = \frac{1}{2}p$ and c_2 from formula (10), after several transformations we obtain /82

$$
\begin{gathered}
(d^2-c^2)\cos^2\frac{\pi p}{2\sigma_T}=a^2+b^2-2c^2-2\sqrt{(a^2-c^2)(b^2-c^2)}\sin\frac{\pi p}{2\sigma_T}; \\
\sqrt{(d^2-a^2)(a^2-c^2)}\left[F(k)+\frac{a^2}{d^2-a^2}\prod\left(-\frac{d^2-c^2}{d^2-a^2},k\right)\right]= \\
=\sqrt{(d^2-b^2)(b^2-c^2)}\left[F(k)+\frac{b^2}{d^2-b^2}\prod\left(-\frac{d^2-c^2}{d^2-b^2},k\right)\right].
\end{gathered} \tag{13}
$$

The length of the slip bands is determined by these equations.

The functions (5) have the following form when equation (13) is satisfied

$$
\begin{gathered}
\Phi(z)=\Omega(z)-\frac{1}{2}p= \\
=-\frac{\sigma_T}{\pi i}\ln\frac{[\sqrt{(z^2-c^2)(d^2-a^2)}+\sqrt{(d^2-z^2)(a^2-c^2)}]\sqrt{b^2-z^2}}{[\sqrt{(z^2-c^2)(d^2-b^2)}+\sqrt{(d^2-z^2)(b^2-c^2)}]\sqrt{z^2-a^2}}-\frac{p}{4}.
\end{gathered} \tag{14}
$$

Let us investigate the case when the inner slip bands are combined -- i.e., $c = 0$. We obtain the following directly from the first equation (13)

$$
d=\sec\frac{\pi p}{2\sigma_T}\sqrt{a^2+b^2-2ab\sin\frac{\pi p}{2\sigma_T}}. \tag{15}
$$

The second equation in (13) may be transformed to the following form[1] after expansion of the indeterminate form in the case k = 1:

$$\frac{b}{a}\cdot\frac{d}{b}=\operatorname{ch}\left(\frac{b}{a}\operatorname{arch}\frac{d}{b}\right)=\frac{1}{2}\left[\left(\frac{d}{b}+\sqrt{\frac{d^2}{b^2}-1}\right)^{\frac{b}{a}}+\left(\frac{d}{b}-\sqrt{\frac{d^2}{b^2}-1}\right)^{\frac{b}{a}}\right]. \tag{16}$$

Eliminating $\frac{d}{b}$ from equations (15) and (16), we obtain the relationship for determining the loading $p = p_0$, at which the inner slip bands are combined:

$$\begin{gathered}\frac{b}{a}\sec\frac{\pi p_0}{2\sigma_T}\sqrt{\frac{a^2}{b^2}+1-2\frac{a}{b}\sin\frac{\pi p_0}{2\sigma_T}}=\\=\operatorname{ch}\left[\frac{b}{a}\operatorname{arch}\left(\sec\frac{\pi p_0}{2\sigma_T}\sqrt{\frac{a^2}{b^2}+1-2\frac{a}{b}\sin\frac{\pi p_0}{2\sigma_T}}\right)\right].\end{gathered} \tag{17}$$

A graph showing the dependence of $\frac{p_0}{\sigma_T}$ on $\frac{b-a}{a}$ is shown in Figure 2.

In the case $p \geqq p_0$ (for each given $\frac{b}{a}$) the length of the outer slip bands may be determined by formula (15). It may be shown that equation (16) must thus /83
be replaced by the inequality $\frac{b}{a}\cdot\frac{d}{b} \leqq \operatorname{ch}\left(\frac{b}{a}\operatorname{arch}\frac{d}{b}\right)$ which expresses the condition $v(0, +0) \geqq 0$. This inequality is satisfied identically in the case $p \geqq p_0$.

After the slip bands are combined, the functions (14) have the form

$$\Phi(z)=\Omega(z)-\frac{1}{2}p=-\frac{\sigma_T}{\pi i}\ln\frac{\left[z\sqrt{d^2-b^2}+a\sqrt{d^2-z^2}\right]\sqrt{b^2-z^2}}{\left[z\sqrt{d^2-b^2}+b\sqrt{d^2-z^2}\right]\sqrt{z^2-a^2}}-\frac{1}{4}p. \tag{18}$$

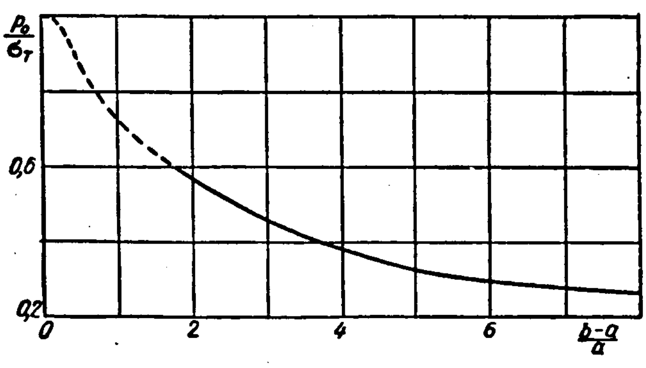

Figure 2

[1] We should note that when $\frac{b}{a}$ is a whole number, the right hand side of (16) is a Chebyshev polynomial of the first kind.

The solution which is obtained pertains to the first stage of elastic-plastic equilibrium of the plate, when the slip bands are propagated only along the abscissa axis. However, for a certain loading, new slip bands arise at the ends of the slits; these slip bands are inclined at an angle to the first ones (they are shown by the dashed line in Figure 1). This is the second stage of elastic-plastic eqiulibrium of the plate. The new slip bands are produced when the maximum shearing stresses $\tau_{max} = 0.5|\sigma_y - \sigma_x + 2i\tau_{xy}|$, which are in operation in the regions perpendicular to the plate plane, reach the flow limit in the vicinity of the slit ends. These stresses may be found from the second formula in (11) substituting the function (14) in it, assuming $z = \pm (a + re^{i\vartheta})$ or $z = \pm (b + re^{i\vartheta})$, and determining the maximum stress as a function of ϑ in the case $r \to 0$. As a result, we obtain

$$\tau_{max} = \frac{1}{2}\left(\frac{\sigma_T}{\pi} + \sqrt{p_2 + \frac{\sigma_T^2}{\pi^2}}\right). \tag{19}$$

On the basis of the formula $\operatorname{ctg} 2\alpha = \frac{2\tau_{xy}}{\sigma_y - \sigma_x}$ as well as formulas (11) and (14), we may show that these stresses influence the regions whose angle of inclination α to the abscissa axis fulfill the following relationship /84

$$\operatorname{tg} 2\alpha = -\pi\frac{p}{\sigma_T} \left(\frac{\pi}{4} < \alpha \leqslant \frac{\pi}{2}\right). \tag{20}$$

Based on the condition of plasticity (1), we find from equation (19) that new slip bands are produced at the loading

$$p_* = \sigma_T\sqrt{1 - \frac{2}{\pi}} \approx 0{,}60\sigma_T. \tag{21}$$

It follows from formulas (20) and (21) that the initial angle of inclination of the new slip bands is

$$\alpha = \frac{1}{2}[\pi - \operatorname{arctg}\sqrt{\pi(\pi - 2)}] \approx 59°. \tag{22}$$

Plate with one slit. Assuming that $a = 0$ in formulas (15) and (18), we may solve the problem of the slip bands when a plate with one slit having the length $2b$ is under tension. In this case, we have

$$d = b \sec\frac{\pi p}{2\sigma_T}; \tag{23}$$

$$\Phi(z) = \Omega(z) - \frac{1}{2}p = -\frac{\sigma_T}{2\pi i}\ln\frac{b\sqrt{d^2 - z^2} - z\sqrt{d^2 - b^2}}{b\sqrt{d^2 - z^2} + z\sqrt{d^2 - b^2}} - \frac{1}{4}p. \tag{24}$$

The loading at which new slip bands appear at the slit ends, and the angle of inclination of these bands, are determined by the formulas (21) and (22).

This solution coincides with the solution given in (Ref. 1, 6). We should note that the development of slip bands when thin plates made of a soft steel with one slit are under tension has been studied experimentally (Ref. 3, 6). Experiments have substantiated the development of plastic deformations arising from the analytical solution. The slip bands first appear on the extension lines of the slit, and new bands directed at an angle to the first ones are produced at a specific loading.

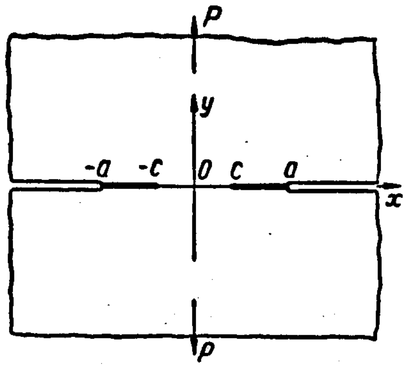

Figure 3

Plate with two semiinfinite cuts. Let us investigate the development of slip bands in a plate with two semi-infinite cuts on the abscissa axis subjected to tension at infinity along the Oy-axis by stresses whose principal vector equals P (Figure 3).

The solution of this problem may be obtained from formulas (5), (7), (9) and (10), assuming that $b = d \to \infty$ and also assuming that the stresses vanish at infinity ($p \to 0$), so that the product pd remains bounded. As a result, we obtain /85

$$\Phi(z) = \Omega(z) = -\frac{\sigma_T}{2\pi i}\left(\frac{2\sqrt{a^2-c^2}}{\sqrt{z^2-c^2}} + \ln\frac{\sqrt{z^2-c^2}-\sqrt{a^2-c^2}}{\sqrt{z^2-c^2}+\sqrt{a^2-c^2}}\right) + \frac{A}{i\sqrt{z^2-c^2}}, \tag{25}$$

where A is a certain constant. This constant may be determined according to the well known (Ref. 4) equation $\lim_{z\to\infty} z\,\Phi(z) = -\frac{iP}{2\pi}$, from which we obtain $A = \frac{P}{2\pi}$.

We obtain the relationship for determining the length of these bands

$$c = a\sqrt{1-\left(\frac{P}{2a\sigma_T}\right)^2}. \tag{26}$$

from the condition of boundedness and continuity of the stress at the end of the slip bands (at the points $z = \pm c$). When this equation is satisfied, the functions (25) assume the form

$$\Phi(z) = \Omega(z) = -\frac{\sigma_T}{2\pi i}\ln\frac{\sqrt{z^2-c^2}-\sqrt{a^2-c^2}}{\sqrt{z^2-c^2}+\sqrt{a^2-c^2}}. \tag{27}$$

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/ M ?

PHOTOELASTIC INVESTIGATION OF STRESS CONCENTRATIONS NEAR A CIRCULAR HOLE IN A HYPERBOLIC SHELL /86

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The purpose of this article is to determine the nature of the stress state in a shell which is a one-sheeted hyperboloid of revolution having constant thickness. This shell is weakened by a small circular hole on the neck under axial compression.

The study employs the photoelasticity method with the use of "freezing" of deformation and subsequent sawing of the hyperboloid model into sections (Ref. 1, 3). The hyperboloid is made of an optically active ED6-M material.

The geometry of the hyperboloid was determined by defining the outer surface

$$x^2 + y^2 - z^2 = R_1^2.$$

The outer diameter of the neck is D_1 = 110 mm; the radius of the middle hyperboloid surface on the neck is R_0 = 52.5 mm; the hyperboloid thickness is h = const = 5 mm; the hole radius is r_0 = 4 mm; and the height of the hyperboloid is H = 116 mm.

The hyperboloid model was compressed with a force of 324 n in a lever device between two steel plates. The freezing temperature and the optical constant of the material at the freezing temperature were determined on a disk whose diameter was compressed and on a band which was extended:

$$T = 116°\mathrm{C}; \quad C = 177 \cdot 10^{-7}\ \mathrm{cm}^2/\mathrm{n}.$$

Figure 1 shows two sets of the sections being studied I, II which are perpendicular to each other. The normal translucence of both sets of sections and the measurements of the corresponding polarization angles ϕ, as well as the difference in the behavior of δ, were determined on a KCP-5 polarization device by means of a Krasnov compensator. /87

Principal stress state. The pictures of the bands during continuous radioscopy of a "frozen" hyperboloid near the hole and far from it indicate that the disturbance of the stress state is localized. An investigation of both sets of sections located far from the hole has substantiated this. Therefore, in view of the axial symmetry of the principal stress state, normal radioscopy of sections I of the set in the direction of the θ axis has provided the difference in, and direction of, the main stresses in the ℓr plane. Normal radioscopy of sections II of the set in the direction of the ℓ axis has given the difference in, and direction of, the principal stresses in the r_θ plane.

Radioscopy of sections in the first set was performed at points of the lines which were normal to the middle surface of the hyperboloid (Figure 1).

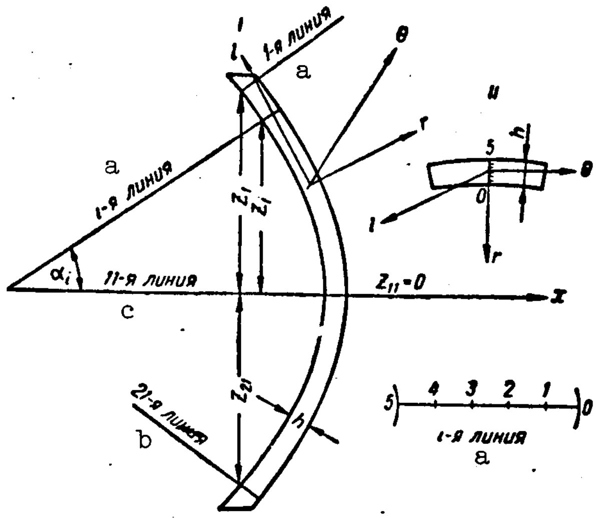

Figure 1

(a) - 1st line; (b) - 21st line;
(c) - 11th line.

There were twenty one lines in each section, and six points, including the contour points, were measured in each line. The parameters of the isoclines along the section lines were constant everywhere, except the lines directly adjacent to the contour (first line and second line), and corresponded to the section geometry. Thus, the parameter of the large principal stress equalled α_i (see Figure 1). Consequently, σ_r and σ_ℓ were the principal stresses along sections I of the set. The stress σ_r was determined on the neck by numerical integration of the second equilibrium equation in cylindrical coordinates (Ref. 2). The maximum value of σ_r did not exceed 3% of σ_ℓ. When the stresses σ_r were disregarded, it was possible to obtain the principal stresses σ_ℓ and σ_θ directly by normal radioscopy of sections I and II of the sets.

The stresses T_ℓ, computed on the basis of the stress σ_ℓ which is thus obtained, satisfy the static condition with an error of 3-5%. /89

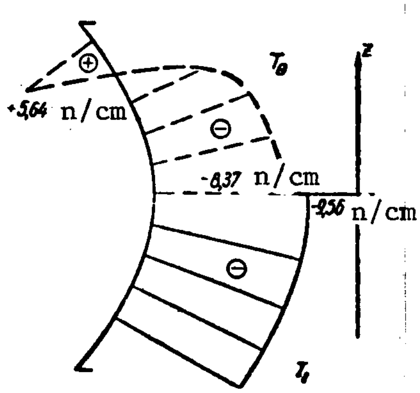

Figure 3

The values of σ_ℓ, σ_θ, as well as the stresses T_ℓ and T_θ computed on the basis of these values, characterize the principal stress state and are given in the table. The corresponding diagrams are shown in Figures 2 and 3.

Stress state around the hole. Two holes at the ends of one diameter are drilled in the neck (z = 0) in the hyperboloid model. The distribution of forces and stresses around the hole may be obtained by interpreting the picture of the bands (Figure 4) obtained by continuous radioscopy of a portion of the frozen model with a hole, and obtained during normal radioscopy of the sections I and II of the set cut out around the hole.

A picture of the bands around the hole shows that the disturbance of the stress field, caused by the hole, is localalized. Figure 5 shows a cross /90

1	2	3								
Номер линии среза	Расстояние z, мм	Напряжение, П/см²	Point Number						T_l, П/см	T_θ, П/см
			0	1	2	3	4	5		
1	50	σ_l	—	—	—	—	—	—	—	5,64
		σ_θ	+11,3	+11,3	+11,3	+11,3	+11,3	+11,3		
3	40	σ_l	—13,5	—14,7	—17	—18,8	—20,7	—22,5	—8,93	—0,44
		σ_θ	+6,0	+3,4	+0,8	—2,65	—5,3	—7,5		
5	30	σ_l	—14,3	—16,0	—17,8	—19,4	—21,1	—22,9	—9,30	—5,88
		σ_θ	—7,5	—8,6	—10,2	—11,3	—12,3	—13,5		
7	20	σ_l	—15,8	—17	—18,2	—19,6	—20,7	—22,2	—9,46	—7,36
		σ_θ	—14,7	—14,7	—14,7	—14,7	—14,7	—14,7		
9	10	σ_l	—15,8	—17,0	—18,4	—19,9	—21,1	—22,9	—9,59	—7,98
		σ_θ	—18,0	—17	—16,2	—15,6	—15,1	—13,9		
11	0	σ_l	—15,8	—16,8	—18,8	—19,6	—20,9	—22,9	—9,56	—8,37
		σ_θ	—19,6	—18,0	—17,4	—16,2	—15,2	—14,1		

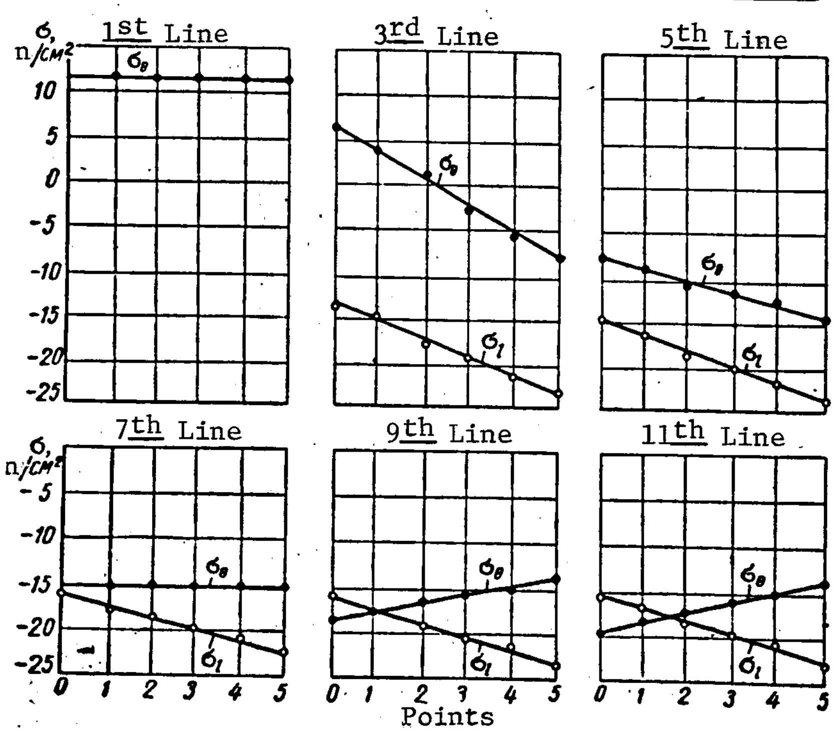

Figure 2

(1) Section line number; (2) distance; (3) stress.

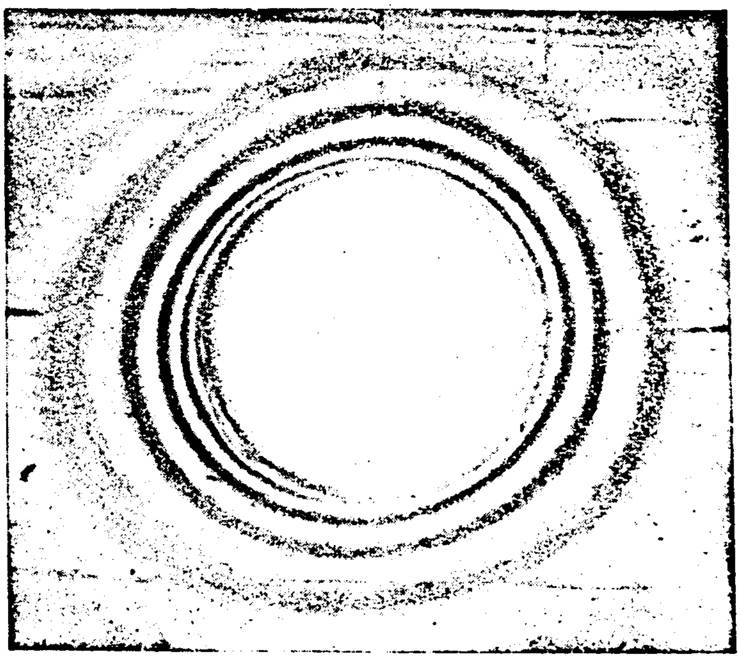

Figure 4

section of the model in the sections I and II with the hole.

Radioscopy of section I enables us to obtain the stress distribution σ_ℓ over the shell thickness along the lines which are located at distances of $z_i' = u_i h$ (h -- shell thickness) from the hole profile. This is shown in Figure 6. Similarily, by employing section II, we may obtain the distribution of σ_θ over the hyperboloid thickness along the lines which are located at distances of $s_i = v_i h$ from the hole profile over the arc of the middle line of the neck. This distribution is shown in Figure 7.

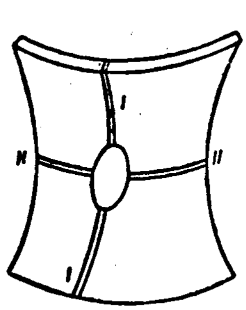

Figure 5

By knowing σ_ℓ over the section I and σ_θ over the section II, we may compute T_ℓ over the section I and T_θ over the section II. If we determine the bands of $T_\ell - T_\theta$ in both sections on the basis of the figure, we may find T_ℓ and T_θ. The distribution of these forces over the sections I and II close to the hole is shown in Figure 8. It may be seen from the graph that even at a distance equalling the hole

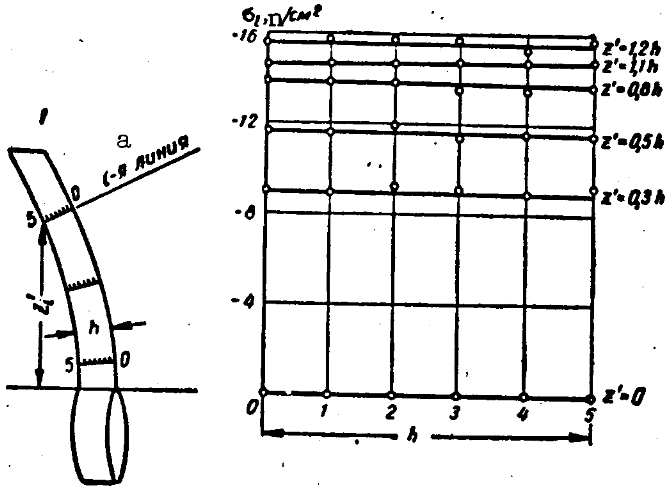

Figure 6

(a) - 1st line

diameter the forces T_ℓ - T_θ along both lines of the main hyperboloid curvature coincide with the principal stress state, within an accuracy of the experiment.

The distribution of the forces along the hole profile is shown in Figure 9. The solid line represents the experimental points obtained from the picture of bands around the hole.

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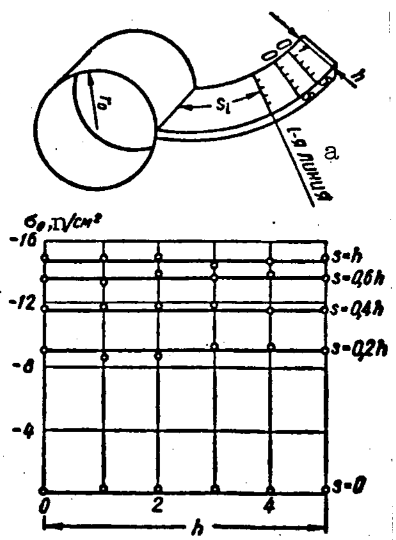

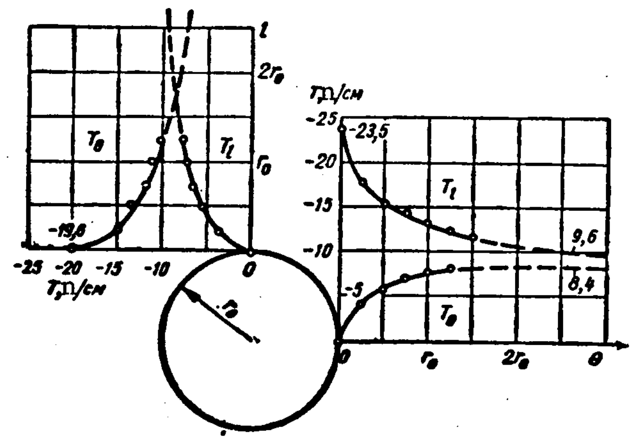

Figure 8

Figure 7

(a) - 1st line

The distribution of forces over the hyperboloid hole profile may be compared with the distribution of forces over the hole profile in a plate which is compressed at infinity by the forces T_ℓ and T_θ corresponding to the principal stress state (Ref. 4). The calculated curve is shown in Figure 9 by the /92

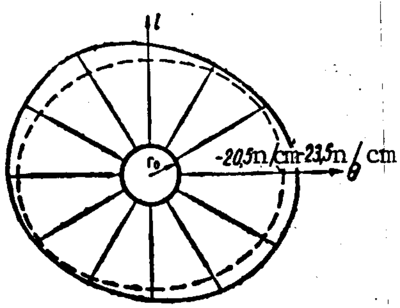

Figure 9

dashed line. The ratios of the maximum force on the hyperboloid hole profile, T^H_{max} and T_θ, of the principal stress state and the maximum force on the hole profile of the plate, $T^{p\ell}_{max}$ and T_θ, equal: $\frac{T^H_{max}}{T_\theta} = 2.88$, $\frac{T^{p\ell}_{max}}{T_\theta} = 2.45$. However, these numerical results must be further refined, since the experiment was performed on only one model.

The following conclusions may be drawn from this experimental investigation of the stress state of a shell having negative Gaussian curvature:

1. The disturbance of the principal stress state of a shell, which is caused by a hole in the shell, is local.

2. The stresses σ_ℓ and σ_θ over the shell thickness change according to a linear law both close to, and far from, the hole.

3. If we know the principal stress state, we may obtain an approximate value for the force concentration. This may be done by comparing the stress state around the hole on the hyperboloid neck with the stress state of a plate loaded at infinity by forces equalling the force of the principal stress state of the shell.

The authors would like to thank academician G. N. Savin of the USSR Academy of Sciences for formulating the problem and for valuable discussion of the results.

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DETERMINATION OF STRESS CONCENTRATION ON THE BASIS OF APPLIED THEORY /93

(Annotation of the Report)

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N67-24511

This report investigates the possibility of employing the equations of applied theory for plate bending, based on the hypothesis of Kirchhoff, for calculating stress concentration. A plate limited by a cylindrical surface Γ_1 and having a hole which is limited by a cylincrical surface Γ_2 is investigated. It is assumed that the hole diameter and the distance between Γ_1 and Γ_2 are large as compared with the plate thickness. It is assumed that the surface Γ_2 and the plane edges of the plate are free from stress, and that the surface Γ_1 is loaded with a certain system of stresses.

The authors base their discussion on relationships given in the work by A. I. Lur'ye[1] which express the stress-deformation state in a solid plate by means of a system of functions satisfying certain differential equations. These relationships include the solution of the same problem according to the Kirchhoff theory as one of their components. The boundary conditions for these functions are determined, based on the Lagrange principle. Representing these functions on the plate contour in series in powers of the plate thickness, the authors obtain recurrent infinite systems of equations, from which all the coefficients of these series are obtained consecutively. At each stage in which the approximations are compiled, we must thus solve the biharmonic problem, which arises in applied theory. We must also solve a certain infinite system of linear algebraic equations -- i.e., we must transform a certain infinite matrix which is the same for all approximations. The elements of this matrix depend neither on the external loading, nor on the boundary contour of the plate.

In order to free Γ_2 from stresses belonging to Γ_1, by employing a similar method we may determine the stress state which disappears when one recedes from Γ_2 and which assumes values for Γ_2 which equal in magnitude, and which have the opposite sign of, the stresses along Γ_1. The total stress state gives the solution of this problem. /94

An investigation of the formulas obtained for stresses on the surface Γ_2 indicated that the error in determining the stress σ_s on Γ_2 according to the applied theory is, with respect to the plate thickness, at least one order of magnitude higher than the stress itself. This conclusion is important, because very frequently the stress concentration coefficient at the hole is determined by the quantity σ_s. The situation is different with the stress τ_{sz}. In the exact solution, it has the first order with respect to the plate thickness, and in applied theory it has the second order. Thus, the applied

[1] A. I. Lur'ye. Prikladnaya Matematika i Mekhanika, Vol. VI, No. 2-3, 1942.

theory distorts the order of the quantity under consideration. Therefore, if the stress concentration coefficient at Γ_2 is not determined on the basis of σ_s, but is rather determined on the basis of any complex characteristic of the stress state containing τ_{sz} (for example, in terms of the Maxwell normal stress), then in this case the use of the applied theory may entail error of the same order with respect to the plate thickness as the quantity characterizing the stress concentration.

/M 3

HINGE-SUPPORTED AND FLEXIBLY CLAMPED SHALLOW SPHERICAL SHELL WITH A HOLE 6 /95

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N67-24512

Complication of the form of the region of the boundary value problem which is solved approximately leads to an increase in the number of unknowns of the approximating system of linear algebraic equations. The efficiency can be retained by compiling the algorithm of increased convergence. A further investigation of the problem reveals that it is possible to reduce them first to integral equations of the appropriate form. The possibility of simplifying the corresponding differential equations is also employed.

The well known system of equations

$$\Delta\Delta\Phi - \frac{Eh}{R}\Delta W = 0; \tag{1}$$

$$\Delta\Delta W + \frac{12(1-\sigma^2)}{Eh^3R}\Delta\Phi = Z, \tag{2}$$

which describes elastic equilibrium of a shallow spherical shell entails the introduction of the following notation, which was given in [Ref. 1, page 399]:

$$W = \Delta\Delta F; \quad \Phi = \frac{Eh}{R}\Delta F$$

only under the condition

$$\Delta\Phi + \frac{Eh}{R}W = 0 \text{ in } \Omega. \tag{3}$$

Here W and Φ are the bending and stress functions; R and h -- radius of curvature of the middle surface and shell thickness; E and σ -- elasticity modulus and Poisson coefficient, respectively; $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ -- two-dimensional Laplace operator.

Let the shell under consideration in the plane of the curvilinear coordinates x_1, x_2 of its middle surface fill the region Ω (possibly a multiply connected region), which is bounded by the profile S. /96

We shall employ the symbols $\frac{\partial}{\partial\tau}$ and $\frac{\partial}{\partial\nu}$ to designate differentiation in the tangent and normal directions, respectively, to S, and shall employ U_τ to designate the displacement of the shell points along the profile. We shall employ the term flexible clamping to designate the fulfillment of the conditions

$$W|_s = U_\tau|_s = \left.\frac{\partial^2\Phi}{\partial\tau^2}\right|_s = 0; \tag{4}$$

$$\left.\frac{\partial W}{\partial\nu}\right|_s = 0, \tag{5}$$

which allow its points in the middle surface to be displaced only normal to the profile.

It follows from condition (4) that $\Delta\Phi|_S = 0$. Cancellation at the boundary S of the function $\Delta\Phi + \frac{Eh}{R} W$ which is harmonic in Ω is expressed by (3). Thus, under condition (4) system (1), (2) is actually reduced to a triangular form with a decrease in the order of magnitude simultaneously:

$$\Delta\Delta W + \lambda^4 W = Z; \tag{6}$$

$$\Delta\Phi + \frac{Eh}{R} W = 0 \quad \left(\lambda^4 = \frac{12(1-\sigma^2)}{R_s h_s}\right). \tag{7}$$

The potential representation

$$\frac{\partial W(x)}{\partial x} = \int_S G(x, \xi)\mu(\xi)\, d_\xi S \tag{8}$$

reduces the problem of the shell flexible clamping (4), (5) to a system of regular integral equations

$$\mu(x) = \varphi(x) - \int_S G(x, \xi)\mu(\xi)\, d_\xi S, \tag{9}$$

which represent direct generalization to the case under consideration of the well known Lauricella-Sherman equations [(Ref. 4), page 385].

Here we have

$$\frac{\partial W}{\partial x} = \begin{pmatrix} \frac{\partial W}{\partial x_1} \\ \frac{\partial W}{\partial x_2} \end{pmatrix}; \quad \mu(x) = \begin{pmatrix} \mu_1(x) \\ \mu_2(x) \end{pmatrix}; \quad \varphi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix};$$

$$G(x, \xi) = L\left(\xi, \frac{\partial}{\partial x}\right)\omega(x, \xi);$$

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$$L\left(\xi, \frac{\partial}{\partial x}\right) = \begin{pmatrix} \frac{\partial^2}{\partial x_1 \partial x_2} & -\frac{\partial^2}{\partial x_1^2} \\ \frac{\partial^2}{\partial x_2^2} & -\frac{\partial^2}{\partial x_2 \partial x_1} \end{pmatrix} \frac{\partial}{\partial \tau} + \begin{pmatrix} \frac{\partial}{\partial \nu}\frac{\partial}{\partial \tau} \\ -\frac{\partial}{\partial \tau}\frac{\partial}{\partial \nu} \end{pmatrix} \frac{\Delta}{2};$$

where $\omega(x, \xi)$ represent the fundamental solution of equation (6).

When the region Ω is bounded by ellipses, it is advantageous to approximate system (9) by replacing the elements of the kernal $G(x\xi)$ by trigonometric polynomials.

For example, let us set $S = S_1 \cup S_2$, where the profiles S_1 and S_2 are determined by the equations

$$x_1 = a\cos t \text{ and } 3x_1 = a\cos t;$$
$$x_2 = a\sin t \text{ and } 4x_2 = a\sin t$$

(elliptical hole in a circular shell with a distance between the profiles equal to the major axis). Let us assume symmetry of the loading Z with respect to both coordinate axes.

In this case, the infinite system of linear algebraic equations, which is formed from (9) by expansion of the kernel in trigonometric series, is a quasiregular system (incidentally, this latter fact is due to the regular nature of the system (9) and does not disappear when there is no system symmetry).

If we set $\frac{12(1 - \sigma^2)a^4}{R^2h^2} = 10^3$, then 12 equations are sufficient for solving system (9) within an accuracy of 1%. Due to the quasiregular nature of the system, it is possible to use an iteration algorithm which reduces the system to four equations.

In the case of the loading

$$Z = \alpha(x_1^2 + x_2^2)$$

we obtain

$$\mu^{(1)} = \begin{pmatrix} -114{,}2888\cos t - 11{,}4670\cos 3t + 1{,}1551\cos 5t \\ 78{,}8615\sin t + 17{,}4682\sin 3t - 0{,}0517\sin 5t \end{pmatrix};$$

$$\mu^{(2)} = \begin{pmatrix} 0{,}0102\cos t + 26{,}7016\cos 3t + 0{,}0058\cos 5t \\ 2{,}9929\sin t - 31{,}8345\sin 3t - 1{,}8551\sin 5t \end{pmatrix}$$

[where ($\mu^{(1)}$ and $\mu^{(2)}$ are the potential densities (8) on the profiles S_1 and S_2, respectively].

This leads us to the conclusion that this method may be applied to obtain an effective solution of several other complex problems (lack of symmetry, a more significant convergence of the profiles, and increase in the amount holes).

The problem of the hinge-supported shallow spherical shell along its profile S is determined by conditions (4) and /98

$$\left(\Delta W + \frac{1-\sigma}{\rho}\cdot\frac{\partial W}{\partial \nu}\right)\Big|_S = 0, \tag{10}$$

where ρ is the radius of curvature of the profile S.

In this case, the system of equations (1), (2) is simplified to the form (6), (7). The potential representation

$$W(x) = \int_{S_1}\left\{\frac{\sigma - 1}{\rho}\cdot\frac{\partial\omega(x,\xi)}{\partial\nu} - \Delta\omega(x,\xi)\right\}\mu_1(\xi)\,d_\xi S - \int_{S_2}\omega(x,\xi)\,\mu_2(\xi)\,d_\xi S$$

reduces the problem (6), (7), (4), (10) to a system of integral equations

$$\begin{aligned} \mu_1(x) &= 2\int_{S_1}\left\{\frac{\sigma-1}{\rho(\xi)}\cdot\frac{\partial^2\omega(x,\xi)}{\partial\nu_x\partial\nu_\xi}-\frac{\partial\Delta\omega(x,\xi)}{\partial\nu}\right\}\mu_1(\xi)\,d_\xi S-\\ &\quad -2\int_{S_1}\frac{\partial\omega(x,\xi)}{\partial\nu}\mu^2(\xi)\,d_\xi S+\varphi_1(x);\\ \frac{\sigma-1}{\rho(x)}\mu(x) &= 2\int_{S_1}\left\{\frac{\sigma-1}{\rho(\xi)}\cdot\frac{\partial\Delta\omega(x,\xi)}{\partial\nu}+\lambda^4\omega(x,\xi)\right\}\mu_1(\xi)\,d_\xi S-\\ &\quad -2\int_{S_1}\Delta\omega(x,\xi)\,\mu_2(\xi)\,d_\xi S+\varphi_2(x), \end{aligned} \tag{11}$$

which are similar, in a certain sense, to the so-called canonical functional equations of V. D. Kupradze (Ref. 3). As has been shown in (Ref. 2, 3), equations (11) may be solved effectively by the method presented above.

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* Translator's Note: Available in English edition.

HINGED CLAMPING OF A SHALLOW SPHERICAL SHELL BOUNDED BY AN ELLIPSE AND RECTANGLE /99

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N67-24513

The study (Ref. 1) has derived an approximate solution for the problem of elastic equilibrium of a shallow spherical shell bounded by a rectangle and by a curve whose parameters are defined -- for example, an ellipse. The conditions of hinged clamping are satisfied in the rectilinear sections of the profile, and these condtions are somewhat simplified in the curvilinear section. These simplifications are removed here. The notation employed in (Ref. 1) is retained, in addition to the notation which is introduced.

The system of differential equations characterizing elastic equilibrium of the shell under consideration may be written as follows:

$$(1-\varkappa)\,\Delta u + 2\varkappa\partial\partial' u = -\frac{4\varkappa}{R}\,\partial w; \tag{1}$$

$$\Delta\Delta w = \frac{12\,(1-\sigma^2)}{Eh^3}\,F - \frac{12}{h^2}\cdot\frac{1+\sigma}{R}\,\partial' u, \tag{2}$$

where

$$\varkappa = \frac{1+\sigma}{3-\sigma};\quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix};\quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix};\qquad \partial = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix};$$

$$\partial' = \left(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2}\right),\quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2};$$

where x_1, x_2 are the curvilinear coordinates of an arbitrary point x of the shell middle surface; $u_1 = u_1(x)$, $u_2 = u_2(x)$, $w = w(x)$ -- vector components of the displacement; E and σ -- Young modulus and elasticity coefficient of the material; R and h -- radius of curvature and shell thickness, respectively; $F = F(x)$ -- external loading which is normal to the surface.

The conditions of hinged clamping of the profile S of the region Ω occupied by the shell have the form /100

$$u_{1|_S} = u_{2|_S} = 0; \tag{3}$$

$$w_{|S} = \left(\Delta w + \frac{1-\sigma}{\rho}\frac{\partial w}{\partial n}\right)_{|S} = 0. \tag{4}$$

Here $\rho = \rho(\xi)$ is the radius of curvature of profile S at its point ξ of the plane x_1Ox_2; the symbol $\frac{\partial}{\partial n}$ designates differentiation in the direction of the outer normal $n = n(\xi) = (n_1, n_2)$ to the profile S.

It is assumed in (Ref. 1) that the following boundary conditions are satisfied

$$u_{1|S} = u_{2|S} = 0; \tag{5}$$
$$w_{|S} = \Delta w_{|S} = 0. \tag{6}$$

The solution u_1, u_2, w of the problem (1), (2), (5), (6) was found in (Ref. 1) in the following order: first from the problem (2), (6), the normal bending of the shell w is expressed by the tangential components of the displacement vector u_1, u_2, and these are then determined from the problem (1), (5). In view of the identical nature of the conditions for problems (2), (3) and (2), (5), in order to solve the problem (1), (2), (3), (4) it is sufficient to replace the solution of the preliminary problem (2), (6) given in (Ref. 1) by the solution of the problem (2), (4).

Let us set $\sin\frac{l\pi}{2}\left(\frac{x_1}{a_1}+1\right) = \varphi_l^1(x_1);\quad \sin\frac{m\pi}{2}\left(\frac{x_2}{a_2}+1\right) = \varphi_m^2(x_2);$

$$G_0(x, \xi) = \sum_{l,m} \frac{16a_1^3a_2^3}{\pi^4(l^2a_2^2+m^2a_1^2)^2}\varphi_l^1(x_1)\varphi_m^2(x_2)\varphi_l^1(\xi_1)\varphi_m^2(\xi_2);$$
$$w_0(x) = \sum_{l,m} \frac{16a_1^3a_2^3F_{lm}}{\pi^4(l^2a_2^2+m^2a_1^2)^2}\varphi_l^1(x_1)\varphi_m^2(x_2),$$

so that $F(x) = \sum_{\ell,m} F_{\ell m}\phi_\ell^1(x_1)\phi_m^2(x_2)$;

$$\Delta G_0(x, \xi) = g_0(x, \xi) = -\sum_{l,m} \frac{4a_1a_2}{\pi^2(l^2a_2^2+m^2a_1^2)}\varphi_l^1(x_1)\varphi_m^2(x_2)\varphi_l^1(\xi_1)\varphi_m^2(\xi_2);$$
$$\Delta w_0(x) = -\sum_{l,m} \frac{4a_1a_2F_{lm}}{\pi^2(l^2a_2^2+m^2a_1^2)}\varphi_l^1(x_1)\varphi_m^2(x_2).$$

The solution of the problem (2), (3), (4) may be represented in the following form by means of the corresponding Green equation

$$w(x) = \bar{w}_0(x) + \int_{S_2} G_0(x, \xi)\frac{\partial \Delta w(\xi)}{\partial n} d_\xi S - \int_{S_2}\left(\frac{\partial G_0(x,\xi)}{\partial n_\xi} + \frac{p(\xi)}{1-\sigma}\Delta G_0(x, \xi)\right)\Delta w(\xi)\, d_\xi S. \tag{7}$$

Here we have /101

$$\bar{w}_0 = \sum_{l,m} \frac{16a_1^3a_2^3\bar{F}_{lm}}{\pi^4(l^2a_2^2+m^2a_1^2)^2}\varphi_l^1(x_1)\varphi_m^2(x_2);$$
$$\bar{F}(x) = \frac{12(1-\sigma^2)}{Fh^3}F - \frac{12}{h^2}\frac{1+\sigma}{R}\partial' u = \sum_{l,m}\bar{F}_{lm}\varphi_l^1(x_1)\varphi_m^2(x_2). \tag{8}$$

In the case $x \to \xi$, the limiting transition leads to a system of integral equations:

$$\bar{w}_0(x(t)) = \int\limits_{S_2} \left\{ \frac{\partial G_0(x(t), \xi(\tau))}{\partial n_\xi} + \frac{\rho(\xi)}{1-\sigma} \Delta G_0(x(t)\,\xi(\tau)) \right\} \lambda(\tau) \frac{dS}{d\tau} d\tau - \\ - \int\limits_{S_2} G_0(x(t), \xi(\tau)) \mu(\tau) d\tau; \tag{9}$$

$$\Delta\bar{w}_0(x(t)) = \frac{\lambda(t)}{2} + \int\limits_{S_2} \frac{\partial \Delta G_0(x(t), \xi(\tau))}{\partial n_\xi} \lambda(\tau) \frac{dS}{\partial\tau} d\tau - \\ - \int\limits_{S_2} \Delta G_0(x(t), \xi(\tau)) \mu(\tau) d\tau, \tag{10}$$

where we employ the following notation:

$$\Delta w(x(t)) = \lambda(t); \quad \frac{\partial \Delta w(x(t))}{\partial n} dS = \mu(t)\, dt,$$

whre $x = x(t)$ or $x_1 = x_1(t)$, $x_2 = x_2(t)$ are the equations of the curvilinear section of the profile $S_2 \subset S$.

Let us set

$$\lambda(t) = \sum_n (\lambda_n^1 \cos nt + \lambda_n^2 \sin nt); \quad \mu(t) = \sum_n (\mu_n^1 \cos nt + \mu_n^2 \sin nt);$$

$$\varphi_l^1(x_1(t))\, \varphi_m^2(x_2(t)) = \frac{1}{\pi} \sum_n (P_{lmn}^1 \cos nt + P_{lmn}^2 \sin nt);$$

$$\frac{\partial}{\partial n} \varphi_l^1(x_1(t))\, \varphi_m^2(x_2(t)) \frac{dS}{dt} = \frac{1}{\pi} \sum_n (T_{lmn}^1 \cos nt + T_{lmn}^2 \sin nt);$$

$$\frac{\rho(x(t))}{1-\sigma} \varphi_l^1(x_1(t))\, \varphi_m^2(x_2(t)) \frac{dS}{dt} = \frac{1}{\pi} \sum_n (\tilde{P}_{lmn}^1 \cos nt + \tilde{P}_{lmn}^2 \sin nt);$$

$$\tilde{T}_{lmn}^i = \tilde{P}_{lmn}^i + \frac{4\alpha_1^2\alpha_2^2}{\pi^2 (l^2\alpha_2^2 + m^2\alpha_1^2)} T_{lmn}^i;$$

$$A_{nk}^{ij} = \sum_{lm} \frac{P_{lmn}^i P_{lmk}^j}{l^2\alpha_2^2 + m^2\alpha_1^2}; \quad B_{nk}^{ij} = \frac{P_{lmn}^i T_{lmk}^j}{l^2\alpha_2^2 + m^2\alpha_1^2};$$

$$\tilde{A}_{nk}^{ij} = \sum_{lm} \frac{4\alpha_1^2\alpha_2^2 P_{lmn}^i P_{lmk}^j}{\pi^2 (l^2\alpha_2^2 + m^2\alpha_1^2)^2}; \quad \tilde{B}_{nk}^{ij} = \sum_{lm} \frac{P_{lmn}^i \tilde{T}_{lmk}^j}{l^2\alpha_2^2 + m^2\alpha_1^2};$$

$$q_n^i = \sum_{lm} \frac{P_{lmn}^i \bar{F}_{lm}}{l^2\alpha_2^2 + m^2\alpha_1^2}; \quad \tilde{q}_n^i = \sum_{lm} \frac{4\alpha_1^2\alpha_2^2 P_{lmn}^i \bar{F}_{lm}}{\pi^2 (l^2\alpha_2^2 + m^2\alpha_1^2)^2}; \quad /102$$

$$\mu = \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix}; \quad \lambda = \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix}; \quad q = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix}; \quad \tilde{q} = \begin{pmatrix} \tilde{q}^1 \\ \tilde{q}^2 \end{pmatrix};$$

where A^{ij}, $\tilde{A}^{ij}$, B^{ij}, $\tilde{B}^{ij}$, μ^i, λ^i, q^i, $\tilde{q}^i$ are the matrices and columns of the elements A_{nk}^{ij}, $\tilde{A}_{nk}^{ij}$, B_{nk}^{ij}, $\tilde{B}_{nk}^{ij}$, μ_n^i, λ_n^i, q_n^i, q_n^i, respectively.

By means of this notation, the system of integral equations (9), (10) may be reduced to a system of linear algebraic equations:

$$
\begin{aligned}
A\mu + B^{*}\lambda &= -q; \\
\tilde{A}\mu + \tilde{B}\lambda &= \tilde{q},
\end{aligned}
\tag{11}
$$

where

$$
B^{*} = B + \frac{\pi^{2}}{8a_1 a_2} J;
$$

$$
J = \begin{pmatrix} 1 & 0 & 0 & . & . & . \\ 0 & 1 & 0 & . & . & . \\ 0 & 0 & 1 & . & . & . \\ . & . & . & . & . & . \end{pmatrix}.
$$

The solution μ, λ of system (11) may be obtained in the following form

$$
\lambda = D^{-1}(\tilde{q} + \tilde{A}A^{-1}q); \tag{12}
$$

$$
\mu = -A^{-1}B^{*}D^{-1}(\tilde{q} + \tilde{A}A^{-1}q) - A^{-1}q, \tag{13}
$$

where D^{-1} is the inverse matrix to $D = \tilde{B} - \tilde{A}A^{-1}B^{*}$.

We finally obtain from (7)

$$
\begin{aligned}
w_{lm} = {} & \frac{16a_1^3 a_2^3 F_{lm}}{\pi^4 (l^2 a_2^2 + m^2 a_1^2)^2} + \sum_n \frac{16a_1^3 a_2^3}{\pi^4 (l^2 a_2^2 + m^2 a_1^2)^2} (P^1_{lmn}\mu^1_n + P^2_{lmn}\mu^2_n) + \\
& + \sum_n \frac{4a_1 a_2}{\pi^2 (l^2 a_2^2 + m^2 a_1^2)} (\tilde{T}^1_{lmn}\lambda^1_n + \tilde{T}^2_{lmn}\lambda^2_n).
\end{aligned}
\tag{14}
$$

Writing formulas (12) and (13) in the following form

$$
\begin{aligned}
\lambda^i_n &= \sum_{k,j} (L^{ij}_{nk} q^j_k + \tilde{L}^{ij}_{nk}\tilde{q}^j_k); \\
\mu^i_n &= -\sum_{kj} (M^{ij}_{nk} q^j_k + \tilde{M}^{ij}_{nk}\tilde{q}^j_k)
\end{aligned}
\tag{15}
$$

and taking into account the corresponding expressions for q^i_n and $\tilde{q}^i_n$, we obtain

$$
\begin{aligned}
\lambda^i_n &= \sum_{kj} \left\{ L^{ij}_{nk} \sum_{lm} \frac{P^j_{lmk}\bar{F}_{lm}}{l^2 a_2^2 + m^2 a_1^2} + \tilde{L}^{ij}_{nk} \sum_{lm} \frac{4a_1^2 a_2^2 P^j_{lmk}\bar{F}_{lm}}{\pi^2 (l^2 a_2^2 + m^2 a_1^2)^2} \right\}; \\
\mu^i_n &= -\sum_{kj} \left\{ M^{ij}_{nk} \sum_{lm} \frac{P^j_{lmk}\bar{F}_{lm}}{l^2 a_2^2 + m^2 a_1^2} + \tilde{M}^{ij}_{nk} \sum_{lm} \frac{4a_1^2 a_2^2 P^j_{lmk} F_{lm}}{\pi^2 (l^2 a_2^2 + m^2 a_1^2)^2} \right\}.
\end{aligned}
$$

The final solution of the problem (1), (5) may be obtained according to (Ref. 2), from formulas (17.2) - (20.2) given in the second portion of (Ref. 1) -- namely: /103

$$
\begin{aligned}
u_{i,q,lm} &= \frac{4a_1^2 a_2^2}{\pi^2 (l^2 a_2^2 + m^2 a_1^2)} \left(\frac{\Phi^{*}_{i,q-1,lm}}{a_1 a_2} - \Phi_{i,q-1lm} \right); \\
\Phi^{*}_{i,q,l_m} &= \sum_{kn} N_{knlm}\Phi_{i,q,kn}; \quad \left(\frac{R^{11} \mid R^{12}}{R^{21} \mid R^{22}} \right) = A^{-1}; \\
N_{knlm} &= \sum_{ijrs} \frac{a_1 a_2 P^i_{lms} R^{ij}_{sk} P^j_{knr}}{k^2 a_2^2 + n^2 a_1^2};
\end{aligned}
\tag{16}
$$

$$\Phi_{1q,lm} = \frac{\pi^2(l^2a_2^2 + m^2a_1^2)}{4a_1a_2} u_{1q,lm} - \sum_{kn} \frac{kn\pi^2}{2a_1a_2} \gamma_{kl}(a_1)\gamma_{nm}(a_2) u_{2q,kn} -$$
$$- \frac{2}{R} \sum_{k} \frac{k\pi}{a_1} \gamma_{kl}(a_1) w_{qkm};$$
$$\Phi_{2qlm} = \frac{\pi^2(l^2a_2^2 + m^2a_1^2)}{4a_1^2a_2^2} u_{2q,lm} - \sum_{kn} \frac{kn\pi^2}{2a_1a_2} \gamma_{kl}(a_1)\gamma_{nm}(a_2) u_{1q,kn} -$$
$$- \frac{2}{R} \sum_{k} \frac{k\pi}{a_2} \gamma_{km}(a_2) w_{q,lk}; \qquad (16)$$
$$w_{qlm} = \sum_{kn} G_{knlm}\bar{F}_{qkn}; \quad \bar{F}_{0kn} = \frac{12(1-\sigma^2)}{Eh^3} F_{kn};$$
$$\bar{F}_{q,lm} = -\frac{12(1+\sigma)}{Rh^2}(u^*_{1q,lm} + u^*_{2q,lm}) \quad (q > 1);$$
$$u_i = \sum_{q=1,2,\ldots} \varkappa^q u_{iq}; \quad w = \sum_q \varkappa^q w_q;$$
$$u^*_{iq;km} = \sum \frac{l\pi}{2a_1} u_{ilm}\gamma_{lk} |a_1|; \quad \cos\frac{l\pi}{2}\left(\frac{x_i}{a_i}+1\right) =$$
$$= \sum_k \gamma_{lk}(d_i) \sin\frac{k\pi}{2}\left(\frac{x_i}{a_i}+1\right).$$

Just as in (Ref. 1), the convergence of the series $u\sum_q \kappa^q u_q$ shown in (Ref. 2) in the case $\kappa < 1$, $R = \infty$ leads to the convergence of the specific formula (16) of the algorithm of the successive approximations in the case under consideration $\kappa = \frac{1+\sigma}{3-\sigma} \ll 1$, under the condition that the shell is sufficiently shallow.

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LIMITING EQUILIBRIUM OF A PLATE WEAKENED BY INTERNAL SHARP-POINTED NOTCHES /104

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Let us investigate an infinite plate weakened by a hole in the form of a hypocycloid (astroid, hypocycloid with three branches), stretched at infinity by the stresses p which increase monotonically and which are directed at the angle α to the abscissa axis (see figure).

The function mapping the exterior of the regions bounded by the given profiles onto the right halfplane of the region of the complex variable $s = \sigma + i\tau$ has the following form

$$z = \omega(s) = R\left[\frac{s+1}{s-1} + \frac{1}{n}\left(\frac{s-1}{s+1}\right)^n\right], \tag{1}$$

where $R = \frac{n}{n+1}a$; in the case n = 2 we obtain the mapping of the region bounded by the hypocycloid with three branches, and in the case n = 3 we obtain the mapping of the region bounded by the astroid. With this mapping, the notch profiles cross the imaginary axis 0τ of the complex plane S.

In both cases, we have profiles with cuspidal points, and the function is $\omega'(s) = 0$ at those points of the axis 0τ which correspond to the cuspidal points.

As is well known, in these cases we cannot employ the method of Muskhelishvili (Ref. 3) to determine the elastic equilibrium of the body directly. Therefore, let us employ the method advanced by S. M. Belonosov (Ref. 2).

Let us write the boundary condition of the first main problem in terms of Muskhelishvili stress functions (Ref. 1);

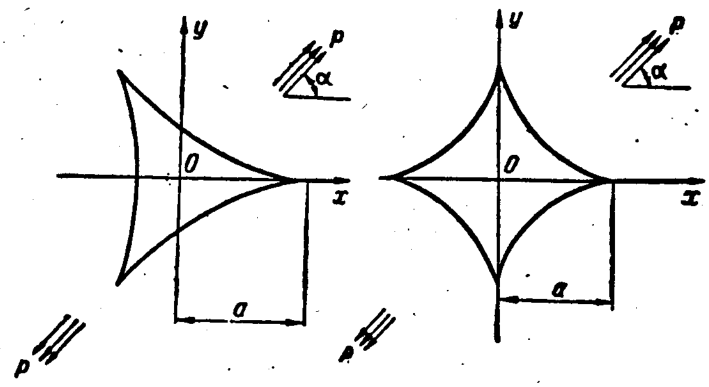
$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = 0. \tag{2}$$

Taking into account tension of the plate at points which are infinitely removed, we may write the functions $\phi(z)$ and $\psi(z)$ as follows: /105

$$\varphi(z) = \frac{p}{4}z + \varphi_0(z);$$
$$\psi(z) = -\frac{p}{2}ze^{-2i\alpha} + \psi_0(z), \tag{3}$$

where $\phi_0(z)$, $\psi_0(z)$ are functions which are regular outside of the hole profile and are bounded at infinity. Conditions (2) may now be written in the following form, with allowance for (3):

$$\varphi_0(z) + z\overline{\varphi_0'(z)} + \overline{\psi_0(z)} = -\frac{p}{2}(z - e^{2i\alpha}\bar{z}). \tag{4}$$



After mapping of the given regions on the right halfplane Res > 0, we obtain the following from (4) by means of expression (1):

$$\varphi_0(\omega(i\tau)) + \frac{\omega(i\tau)}{\overline{\omega'(i\tau)}}\overline{\varphi_0'(\omega(i\tau))} + \overline{\psi_0(\omega(i\tau))} = -\frac{p}{2}[\omega(i\tau) - \overline{\omega(i\tau)}\, e^{2i\alpha}].$$

Let us introduce the notation

$$\begin{aligned} &\Phi(s) = \varphi_0(\omega(s));\\ &\Psi(s) = \frac{\overline{\omega(s)}}{\omega'(s)}\varphi_0'(\omega(s)) + \psi_0(\omega(s)) + \frac{pa}{2}(1 - e^{-2i\alpha});\\ &f_1(\tau) + if_2(\tau) = -\frac{p}{2}[\omega(i\tau) - e^{2i\alpha}\overline{\omega(i\tau)}] + \frac{pa}{2}(1 - e^{2i\alpha}). \end{aligned} \tag{5}$$

In this case, the boundary condition (4) may be written as follows:

$$\Phi(i\tau) + \overline{\Psi(i\tau)} = f_1(\tau) + if_2(\tau). \tag{6}$$

The functions $\Phi(s)$, $\Psi(s)$ are regular in the right halfplane. We shall assume that they are continuous on the imaginary axis; $\Phi(\infty) = \Psi(\infty) = 0$, since they contain an additive constant.

According to the Harnack theorem (Ref. 3), condition (6) is equivalent to the two following relationships: /106

$$\Phi(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f_1(\tau) + if_2(\tau)}{s - i\tau}d\tau; \tag{7}$$

$$\Psi(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f_1(\tau) - if_2(\tau)}{s - i\tau}d\tau. \tag{8}$$

Let us now write the functions $\Phi(s)$ and $\Psi(s)$ for an infinite plane with a notch in the form of a hypocycloid with three branches. For this region, according to formula (1) we may readily find the requisite mapping function

(in the case n = 2)

$$z = \omega(s) = \frac{2a}{3}\left[\frac{s+1}{s-1} + \frac{1}{2}\left(\frac{s-1}{s+1}\right)^2\right]. \tag{9}$$

If we substitute the values of $f_1 + if_2$, $f_1 - if_2$ determined according to formulas (5) and (9) in expressions (7) and (8), and if we then calculate the requisite integrals by means of the theorem of residues, as a result we obtain the following stress functions:

$$\Phi(s) = \frac{2pa}{3(s+1)}\left[\frac{s}{1+s} - e^{2i\alpha}\right]; \tag{10}$$

$$\Psi(s) = \frac{2pa}{3(s+1)}\left[1 + \frac{s-1}{(s+1)^2} - \frac{se^{-2i\alpha} + e^{2i\alpha}}{s+1}\right]. \tag{11}$$

On the basis of formulas (5), (10), (11), we may readily find the Muskhelishvili stress functions $\phi'(z)$, $\psi'(z)$. By means of the well known formulas given in (Ref. 3), we may then find the stress functions at any point on the plate. Let us write the expression $\phi'(z)$ which is requisite for further computations

$$\varphi'(z) = \frac{p}{4} - \frac{p}{4}\cdot\frac{(s-1)^2(s+1)}{1+3s^2}\left[\frac{1-s}{1+s} + e^{2i\alpha}\right]. \tag{12}$$

The stresses in the plate will be greatest in the vicinity of the hole points (the limiting equilibrium of the plate will be determined them).

It may be seen from the studies (Ref. 4, 5) that the coefficients of stress concentration and stress in the vicinity of points such as cuspidal points are determined by the following relationships in the case of two-dimensional tension of a plate with a hole:

$$\begin{gathered}
4\mathrm{Re}\varphi_i'(r,\beta) = k_{1,i}\sqrt{\frac{2}{r}}\cos\frac{\beta}{2} - k_{2,i}\sqrt{\frac{2}{r}}\sin\frac{\beta}{2} + O(r^{-1/2});\\
\sigma_r = \frac{1}{4\sqrt{2r}}\left\{k_{1,i}\left[5\cos\frac{\beta}{2} - \cos\frac{3\beta}{2}\right] + k_{2,i}\left[-5\sin\frac{\beta}{2} + 3\sin\frac{3\beta}{2}\right]\right\} + \\
+ O(r^{-1/2});
\end{gathered} \tag{13}$$

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$$\begin{gathered}
\sigma_\beta = \frac{1}{4\sqrt{2r}}\left\{k_{1,i}\left[3\cos\frac{\beta}{2} + \cos\frac{3\beta}{2}\right] - 3k_{2,i}\left[\sin\frac{\beta}{2} + \sin\frac{3\beta}{2}\right]\right\} + O(r^{-1/2});\\
\sigma_{r\beta} = \frac{1}{4\sqrt{2r}}\left\{k_{1,i}\left[\sin\frac{\beta}{2} + \sin\frac{3\beta}{2}\right] + k_{2,i}\left[\cos\frac{\beta}{2} + 3\cos\frac{3\beta}{2}\right]\right\} + O(r^{-1/2})\\
(i = 1, 2, 3),
\end{gathered} \tag{14}$$

where $k_{1,i}$, $k_{2,i}$ are the coefficients of stress concentration. They depend on the hole configuration and the loading; ϕ_i' $(r\beta)$ -- the function $\phi'(z)$ pertaining to the polar coordinate system (r, β) with the origin at the point $z_i/z_1 = a$, $z_2 = \frac{a}{2} + \frac{ai\sqrt{3}}{2}$, $z_3 = -\frac{a}{2} - \frac{ai\sqrt{3}}{2}$. The orientation of the system may be selected so that the polar axis r_i is directed clockwise with respect to the tangent to the curve at a given point.

In the vicinity of the point $z = a$, the function ϕ_i' (r, β) may be represented as follows:

$$\varphi_1'(r, \beta) = -\frac{p}{b}\sqrt{\frac{a}{r}}e^{-\frac{i\beta}{2}}(-1 + e^{2i\alpha}) + O(r^{1/2}).$$

The coefficients $K_{1,1}$ and $k_{2,1}$ may be determined by comparing the left and right hand sides of equation (13). They have the following form

$$\begin{aligned} k_{1,1} &= \frac{4p\sqrt{a}}{3\sqrt{2}}\sin^2\alpha; \\ k_{2,1} &= \frac{2p\sqrt{a}}{3\sqrt{2}}\sin 2\alpha. \end{aligned} \tag{15}$$

In the vicinty of the point $z_2 = -\frac{a}{2} + \frac{ai\sqrt{3}}{2}$ we have:

$$\begin{aligned} k_{1,2} &= \frac{p\sqrt{a}}{\sqrt{6}}\left(\frac{2}{\sqrt{3}} + \sin 2\alpha + \frac{1}{\sqrt{3}}\cos 2\alpha\right); \\ k_{2,2} &= \frac{p\sqrt{a}}{\sqrt{6}}\left(\cos 2\alpha - \frac{1}{\sqrt{3}}\sin 2\alpha\right), \end{aligned} \tag{16}$$

and in the vicinity of the point $z_3 = -\frac{a}{2} - \frac{ai\sqrt{3}}{2}$, we have:

$$\begin{aligned} k_{1,3} &= \frac{p\sqrt{a}}{\sqrt{6}}\left(\frac{2}{\sqrt{3}} - \sin 2\alpha + \frac{1}{\sqrt{3}}\cos 2\alpha\right); \\ k_{2,3} &= -\frac{p\sqrt{a}}{\sqrt{6}}\left(\cos 2\alpha + \frac{1}{\sqrt{3}}\sin 2\alpha\right). \end{aligned} \tag{17}$$

The expressions (15), (16), (17) may be written in a more compact form as follows:

$$\begin{aligned} k_{1,i} &= \frac{4p\sqrt{a}}{3\sqrt{2}}\sin^2(\alpha - \omega_i); \\ k_{2,i} &= \frac{2p\sqrt{a}}{3\sqrt{2}}\sin 2(\alpha - \omega_i) \quad (i = 1, 2, 3); \\ \omega_1 &= 0; \quad \omega_2 = \frac{2\pi}{3}; \quad \omega_3 = -\frac{2\pi}{3}. \end{aligned} \tag{18}$$

On the basis of equations (14) and (18), we may readily determine the stresses σ_r, σ_β, $\sigma_{r,\beta}$ in the vicinity of the hypocycloid apex. /108

As is well known (Ref. 1), the limiting equilibrium state in a plate with sharp-pointed notches sets in when the propagation of cracks is possible in even one of the notch apexes, under given external stresses. If it is assumed that the initial direction of crack propagation will be along lines where the normal stresses $\sigma_\beta(r, \beta)$ in the vicinity of a point reach maximum intensity, then -- according to the results presented in (Ref. 1, 4) - we obtain the following conditions for determining the magnitude of the limiting stresses:

$$\lim_{r\to 0}\left\{\sqrt{r}\,\frac{\partial\sigma_\beta(r,\beta)}{\partial\beta}\right\}_{\beta=\beta_{*i}} = 0; \tag{19}$$

$$\lim_{r\to 0}\left\{\sqrt{r}\,\sigma_\beta(r,\beta_{*i},p_{*i})\right\} = \frac{K}{\pi}, \tag{20}$$

where K is the material constant (bonding modulus) (Ref. 1).

With allowance for (14), condition (19) may be reduced to the following form (Ref. 4):

$$\begin{aligned}\beta_{*i} &= \pm 2\arcsin\sqrt{\frac{6n_i+1-\sqrt{8n_i^2+1}}{2(9n_i^2+1)}} \quad \text{for } k_{1,i}>0;\\ \beta_{*i} &= \pm 2\arcsin\sqrt{\frac{6n_i^2+1+\sqrt{8n_i^2+1}}{2(9n_i^2+1)}} \quad \text{for } k_{1,i}<0.\end{aligned} \tag{21}$$

In these formulas, the sign + corresponds to $K_{2,i} < 0$, and the sign - corresponds to the value $k_{2,i} > 0$

$$n_i = \frac{k_{2,i}}{k_{1,i}}. \tag{22}$$

For the problem under consideration, we have

$$n_i = \operatorname{ctg}(\alpha-\omega_i). \tag{22}$$

On the basis of equations (20), (14), (18) and (21), we obtain the following formula for determining the critical loading p_{*i}

$$p_{*i} = \frac{3\sqrt{2}}{4}\,\frac{4\sqrt{2}K}{\pi\sqrt{a}\sin(\alpha-\omega_i)\left[3\sin\left(\alpha-\omega_i-\frac{\beta_{*i}}{2}\right)+2\sin\left(\alpha-\omega_i-\frac{3\beta_{*i}}{2}\right)-\sin\left(\alpha-\omega_i+\frac{3\beta_{*i}}{2}\right)\right]} \tag{23}$$

The value

$$p_* = \min\{p_{*i}\} \tag{24}$$

will determine the limiting equilibrium state of the plate.

In the special case when $\alpha = \frac{\pi}{2}$, at the point $z_1 = a$, the crack begins to /109
propagate under the loading $p_{*1} = 0.477\,\frac{K}{\sqrt{a}}$ at the angle $\beta_{*1} = 0$. The crack proceeds from the other crack apexes under large loading $p_{*2} = p_{*3} = 0.735\,\frac{K}{\sqrt{a}}$ at the angles $\beta_{*2} = -60°$, $\beta_{*3} = 60°$. This means that the loading p_{*1} will be the limiting load for the plate.

According to the Griffith formula for a plate with a rectilinear crack having the length 2ℓ subjected to tension at infinity by the stresses p perpendicular to the crack, the critical loading is

$$p_* = 0{,}450 \frac{K}{\sqrt{l}}. \tag{25}$$

In our case, if we assume that the crack length is $2\ell = \frac{4a}{3}$, we find that

$$p_* = 0{,}389 \frac{K}{\sqrt{l}}. \tag{26}$$

Let us now investigate the problem of determining the limiting equilibrium state of a plate with a hole in the form of an astroid. Assuming that n = 3 in formula (1) for this purpose, we obtain the mapping function

$$z = \omega(s) = \frac{3a}{4}\left[\frac{s+1}{s-1} + \frac{1}{3}\left(\frac{s-1}{s+1}\right)^3\right]. \tag{27}$$

In the same way as above, we obtain the functions

$$\Phi(s) = \frac{3pa}{4}\left[\frac{1-e^{2i\alpha}}{s+1} - \frac{2}{(s+1)^2} + \frac{4}{3(s+1)^3}\right]; \tag{28}$$

$$\Psi(s) = \frac{pa}{4(s+1)}\left[3 - \frac{e^{-2i\alpha}(1+3s^2)}{(s+1)^3}\right]; \tag{29}$$

$$\varphi'(z) = \frac{p}{4} - \frac{p(s^2-1)^2}{16s(s^2+1)}\left[e^{2i\alpha} - 1 + \frac{4}{s+1} - \frac{4}{(s+1)^2}\right]. \tag{30}$$

The coefficients of stress concentration in the vicinity of four apexes of the astroid $z_1 = a$, $z_2 = ia$, $z_3 = -a$, $z_4 = -ia$ are determined according to formulas (13) and (30) and have the following form

$$\left.\begin{aligned} k_{1,i} &= \frac{p}{2}\sqrt{3a}\sin^2(\alpha-\omega_i); \\ k_{2,i} &= \frac{p}{4}\sqrt{3a}\sin^2(\alpha-\omega_i) \end{aligned}\right\} \quad (i = 1, 2, 3, 4);$$

$$\omega_1 = 0, \quad \omega_2 = \frac{\pi}{2}, \quad \omega_3 = \pi, \quad \omega_4 = -\frac{\pi}{2}. \tag{31}$$

The initial directions β_{*i} of the crack propagation from the notch points may be determined according to formulas (21) in the case

$$n_i = \mathrm{ctg}(\alpha - \omega_i) \quad (i = 1, 2, 3, 4).$$

Finally, the formula determining the critical loading for each astroid apex is as follows: /110

$$p_{*i} = \frac{2}{\sqrt{3}} \frac{4\sqrt{2}K}{\pi\sqrt{a}\sin(\alpha-\omega_i)\left[3\sin\left(\alpha-\omega_i-\frac{\beta_{*i}}{2}\right) + 2\sin\left(\alpha-\omega_i-\frac{3\beta_{*i}}{2}\right) - \sin\left(\alpha-\omega_i+\frac{3\beta_{*i}}{2}\right)\right]}. \tag{32}$$

When $\alpha = \frac{\pi}{2}$, we have

$$p_* = 0{,}519 \frac{K}{\sqrt{a}} \quad (\beta_* = 0), \tag{33}$$

In this case, we must assume $a = \ell$ in order to compare the result obtained with the Griffith formula.

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INVESTIGATION OF THE STRESS STATE OF SPHERICAL SHELLS WITH MULTIPLY CONNECTED REGIONS /111

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Several articles have investigated the stress state of spherical shells in the case of multiply connected regions. The study (Ref. 3) pointed out methods for an approximate solution of the problems for non-shallow spherical shells having several holes. However, no allowance was made for the influence of combined stress states, which may significantly affect the result when the holes converge. The studies (Ref. 8-10) propose the method of successive approximations to study the stress state in shallow spherical shells in the case of multiply connected regions. The authors comfined themselves to only the first approximation, which made it possible to determine the distance at which the holes had no influence upon each other.

This work plans to investigate the problem of the stress state of shallow spherical shells in the case of multiply connected regions by having the boundary conditions exactly fulfill an infinite system of algebraic equations, which may be written in explicit form[1].

Let us investigate the stress state of a shallow spherical shell, which occupies the (m + 1) - connected region S (Figure 1) bounded by the profile $L = L_1 + \dots + L_m + L_0$ of the circle L_k, where L_0 is the outer profile. We shall connect the coordinate system (x_k, y_k) with the center of each circle L_k; the coordinate system (x, y) is connected with the center of the circle L_0. Under the condition that normal loading alone is present, the investigation /112 of the stress state may be reduced to solving the equation

$$\nabla^2\nabla^2\Phi - i\varkappa^2\nabla^2\Phi = \frac{qr_0^4}{D}. \qquad (1)$$

For the boundary conditions we have

$$L_k^{(t)}\Phi|_{L_k} = f_{kt}(\theta_k)\begin{cases} t = 1, 2, 3, 4;\ z = x + iy;\ z = re^{i\theta}; \\ k = 0, 1, \dots, m;\ z_k = x_k + iy_k;\ z_k = r_k e^{i\theta_k}, \end{cases} \qquad (2)$$

where $\Phi = w + ig\phi$; D -- cylindrical rigidity; q -- intensity of normal loading; all the coordinates are dimensionless, pertaining to r_0, and are interrelated by means of the relationship $z = z_k + \ell_k$; $L_k^{(t)}$ -- differential operators of the boundary conditions;

$$g = \frac{\sqrt{12(1-\nu^2)}}{Eh^2}; \quad \varkappa = r_0 \sqrt[4]{\frac{12(1-\nu^2)}{R^2h^2}}.$$

[1] We shall only investigate circular holes. However, by employing the results given in (Ref. 4, 11) our discussion may be generalized to the case of non-circular holes.

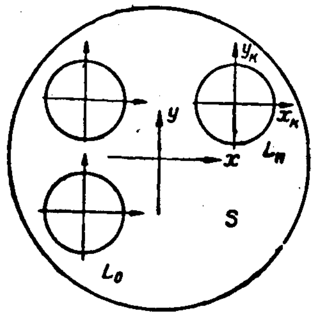

Figure 1

We may write the solution of equation (1) in the form of the sum

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3, \tag{3}$$

where Φ_1 is the solution of the Laplace equation; Φ_2 -- solution of the Helmholtz equation; Φ_3 -- particular solution;

$$\Phi_1 = \sum_{k=1}^{m} iB_k \ln (z - l_k)\overline{(z - l_k)} + \varphi(z) + \overline{\psi(z)}, \quad \operatorname{Im} B_k = 0, \tag{4}$$

where $\phi(z)$ and $\psi(z)$ are functions which are holomorphic in S and which, according to (Ref. 13), may be represented in the following form

$$\varphi(z) = \sum_{k=1}^{m}\sum_{p=1}^{\infty} \frac{\alpha_{cr}}{(z-l_k)^p} + \sum_{p=0}^{\infty} \beta_p z^p; \quad \psi(z) = \sum_{k=1}^{m}\sum_{p=1}^{\infty} \frac{\alpha^{*}_{cr}}{(z-l_k)^p} + \sum_{p=0}^{\infty} \beta^{*}_p z^p. \tag{5}$$

We should point out that functions such as (5) were employed when investigating the plane problem of elasticity theory for multiply connected regions (Ref. 6, 12).

The solution of the Helmholtz equation for the region S may be written in the following form

$$\Phi_2 = \sum_{k=1}^{m}\sum_{p=0}^{\infty} \begin{pmatrix} a_{kp} + ib_{kp} \\ a^{*}_{kp} + ib^{*}_{kp} \end{pmatrix} H_p^{(1)}(r_k \varkappa \sqrt{-i}) \begin{matrix} \cos p\theta_k \\ \sin p\theta_k \end{matrix} + \sum_{p=0}^{\infty} \begin{pmatrix} a_p + ib_p \\ a^{*}_p + ib^{*}_p \end{pmatrix} J_p(r\varkappa\sqrt{-i}) \begin{matrix} \cos p\theta \\ \sin p\theta \end{matrix}. \tag{6}$$

We may represent the components of the stress and deformed states in the form of the sum of three parts corresponding to Φ_1, Φ_2, and Φ_3 . In order to determine the components of the stress and deformed state corresponding to Φ_1 in the μth coordinate system (r_μ, θ_μ), the following relationships are obtained /113

$$\begin{gathered} S^1_{r_\mu \theta_\mu} + iT^1_{r_\mu} = -\frac{1}{gr_0^2} z_\mu \frac{\varphi''(z) - \psi''(z)}{\bar z_\mu} + \frac{2i}{gr_0^2}\sum_{k=1}^{m} \frac{B_k z_\mu}{(z-l_k)^2 \bar z_\mu}; \\ T^1_{\theta_\mu} = -T^1_{r_\mu}; \quad G^1_{r_\mu} = -D\frac{1-\nu}{r_0^2} \operatorname{Re} z_\mu \frac{\varphi''(z) + \psi''(z)}{\bar z_\mu}; \\ G^1_{\theta_\mu} = -G^1_{r_\mu}; \quad \tilde Q^1_{r_\mu} = -\frac{2}{r_0 r_\mu} G^1_{r_\mu} + D\frac{1-\nu}{r_0^3 r_\mu} \operatorname{Re} z_\mu^2 \frac{\varphi'''(z) + \psi'''(z)}{\bar z_\mu}; \end{gathered} \tag{7}$$

$$u_\mu^1 + iv_\mu^1 = -\frac{1+\nu}{Ehgr_0}\left[\overline{\varphi'(z)} - \overline{\psi'(z)} - i\sum_{k=1}^{m} 2\frac{B_k}{z-l_k}\right]\frac{\bar{z}_\mu}{r_\mu} -$$
$$-\frac{r_0}{R}\frac{\bar{z}_\mu}{r_\mu}\int[\varphi(z)+\psi(z)]\,dz + i\frac{r_0}{R}z\frac{\bar{z}_\mu}{r_\mu}\,\mathrm{Im}(\beta_0+\beta_0^*). \qquad (7)$$

The components of the stress and deformed states corresponding to Φ_2 and Φ_3 in the coordinate system $(r_\mu \theta_\mu)$ may be computed according to the customary formulas for the polar coordinate system (Ref. 2). The displacements corresponding to Φ_2 satisfy the conditions of single-valued displacements. If these conditions are satisfied by displacements corresponding to Φ_3 (these conditions are satisfied for problems regarding stress concentration, since $\Phi_3 \equiv 0$), we obtain the following from the condition of single-valued displacements, taking into account (7):

$$\alpha_{kl} + \alpha_{kl}^* = 0. \qquad (8)$$

In (7) the term $i\frac{r_0}{R}z\frac{\bar{z}_\mu}{r_\mu}\mathrm{Im}(\beta_0 + \beta_0^*)$ corresponds to a rigid body displacement of the shell.

Thus, the components of the stress and deformed states may be determined in any coordinate system $(r_\mu \theta_\mu)$. Using these components to calculate the quantities contained in the boundary conditions (2), we may expand them in Fourier series on the μth profile. When calculating the integrals containing the function Φ_1, we must change to the region of the complex variable, for they may be readily computed according to the theorem of residues. When computing the integrals containing the function Φ_2, we must employ the Graff formula for cylindrical functions (Ref. 1, page 394). It is first possible to expand the function Φ_1 in Laurent series in a ring containing the μth profile. In the case of harmonics at each of the profiles, by setting the coefficients equal to zero we obtain an infinite system of algebraic equations, in which /114 the number of unknowns corresponds to the number of equations.

By way of an example, let us investigate the stress state of a spherical shell weakened by two equal circular holes having the radius r_0, whose centers are located at the distance ℓr_0. The shell is loaded by a uniform internal pressure with the intensity q, and the holes are closed by lids which only transmit the influence of the intersection force (Ref. 7). We shall assume that the principal stress state is momentless. We shall also assume that the holes are located at a distance for which the additional stress which is produced may be described by equations given by the theory of shallow shells. The solution of this problem may be reduced to solving a homogeneous equation (1) for an infinite region S (Figure 2) under specific conditions "at infinity" (Ref. 7)

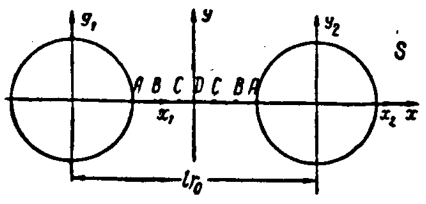

Figure 2

and under the following boundary conditions:

$$T_{r_\mu}\big|_{r_\mu=1} = -p_0 h;\quad S_{r_\mu\theta_\mu}\big|_{r_\mu=1} = 0;\quad G_{r_\mu}\big|_{r_\mu=1} = 0;$$
$$\tilde{Q}_{r_\mu}\big|_{r_\mu=1} = -\frac{qr_0}{2};\quad p_0 = \frac{qR}{2h};\quad \mu = 1, 2. \tag{9}$$

Based on (4) - (6), due to the force and geometric symmetry, we may represent the solution in the following form

$$\Phi_1 = \varphi(z) + \overline{\psi(z)};\ \psi(z) = \bar{\varphi}(z);\ \varphi(z) = \sum_{p=1}^{\infty} \alpha_p\left[\left(z+\frac{l}{2}\right)^{-p} + (-1)^p\left(z-\frac{l}{2}\right)^{-p}\right];\ \Phi_2 = \sum_{p=0}^{\infty}(a_p + ib_p)\times$$
$$\times\,[H_p^{(1)}(r_1\varkappa\sqrt{-i})\cos p\theta_1 + (-1)^p H_p^{(1)}(r_2\varkappa\sqrt{-i})\cos p\theta_2]. \tag{10}$$

We obtain Re $\alpha_1 = 0.3$ from the condition of single-valued displacements (8). We should point that Φ_1 and Φ_2 (10) satisfy the condition "at infinity". Let us write an infinity system of algebraic equations in order to determine a_p, b_p, c_p, and d_p ($\alpha_p = c_p + id_p$):

$$A^t_n a_n + B^t_n b_n + C^t_n c_n + D^t_n d_n + \varepsilon_n \sum_{p=0}^{\infty}(A^t_{np}a_p + B^t_{np}b_p + C^t_{np}c_p +$$
$$+ D^t_{np}d_p) = -p_0\frac{R}{E}\delta^0_n(\delta^1_t + \varkappa\delta^4_t);\ t = 1, 2, 3, 4;\ n = 0, 1, 2, \ldots, \infty;$$
$$\delta^k_n = \begin{cases}1 & n = k;\\ 0 & n \neq k;\end{cases}\quad \varepsilon_n = \begin{cases}\frac{1}{2} & n = 0;\\ 1 & n \neq 0;\end{cases}\ c_0 = 0;\ d_0 = 0;\ c_1 = 0; \tag{11a}$$

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$$C^1_n = 0;\ C^2_n = 0;\ C^1_{np} = 0;\ C^2_{np} = 0;\ D^3_n = 0;\ D^t_n = 0;\ D^3_{np} = 0;$$
$$D^4_{np} = 0. \tag{11b}$$

The following notation is also introduced:

$$A^1_n = \mathrm{her}_n\,\varkappa - \mathrm{hei}''_n\,\varkappa;\ A^2_n = n(\varkappa\,\mathrm{hei}'_n\,\varkappa - \mathrm{hei}_n\,\varkappa);\ A^3_n = (1-\nu)\,\mathrm{her}''_n\,\varkappa -$$
$$-\,\nu\,\mathrm{hei}_n\,\varkappa;\ A^4_n = \mathrm{hei}'_n\,\varkappa + n^2\frac{1-\nu}{\varkappa^3}(\varkappa\,\mathrm{her}'_n\,\varkappa - \mathrm{her}_n\,\varkappa);$$
$$B^1_n = -\mathrm{hei}_n\,\varkappa - \mathrm{her}''_n\,\varkappa;\ B^2_n = n(\varkappa\,\mathrm{her}'_n\,\varkappa - \mathrm{her}_n\,\varkappa);$$

$$B_n^3 = -(1-\nu)\,\mathrm{hei}_n'\,\varkappa - \nu\,\mathrm{her}_n\,\varkappa;\quad B_n^4 = \mathrm{her}_n'\,\varkappa - n^2\frac{1-\nu}{\varkappa^3}(\varkappa\,\mathrm{hei}_n'\,\varkappa - \mathrm{hei}_n\,\varkappa);\quad C_n^3 = 2\frac{1-\nu}{\varkappa^2}\,n(n+1);\quad C_n^4 = -2\frac{1-\nu}{\varkappa^3}\,n^2(n+1);$$

$$D_n^1 = -2\frac{n(n+1)}{\varkappa^2};\quad D_n^2 = -2n(n+1);\quad A_{np}^1 = [\mathrm{her}_{p+n}\,l\varkappa + (-1)^n\,\mathrm{her}_{p-n}\,l\varkappa]\,(\mathrm{ber}_n\,\varkappa - \mathrm{bei}_n'\,\varkappa) - [\mathrm{hei}_{n+p}\,l\varkappa + (-1)^n\,\mathrm{hei}_{p-n}\,l\varkappa] \times (\mathrm{bei}_n\,\varkappa + \mathrm{ber}_n'\,\varkappa);$$

$$B_{np}^1 = -[\mathrm{her}_{p+n}\,l\varkappa + (-1)^n\,\mathrm{her}_{p-n}\,l\varkappa] \times (\mathrm{bei}_n\,\varkappa + \mathrm{ber}_n'\,\varkappa) - [\mathrm{hei}_{p+n}\,l\varkappa + (-1)^n\,\mathrm{hei}_{p-n}\,l\varkappa]\,(\mathrm{ber}_n\,\varkappa - \mathrm{bei}_n'\,\varkappa);$$

$$D_{np}^1 = -\frac{2}{\varkappa^2}\cdot\frac{(p+n-1)!}{(p-1)!\,(n-2)!}\cdot\frac{1}{l^{p+n}};\quad A_{np}^2 = n\,[\mathrm{her}_{p+n}\,l\varkappa + (-1)^n\,\mathrm{her}_{p-n}\,l\varkappa]\,(\varkappa\,\mathrm{bei}_n'\,\varkappa - \mathrm{bei}_n\,\varkappa) + n\,[\mathrm{hei}_{p+n}\,l\varkappa + (-1)^n\,\mathrm{hei}_{p-n}\,l\varkappa]\,(\varkappa\,\mathrm{ber}_n\,\varkappa - \mathrm{ber}_n\,\varkappa);$$

$$B_{np}^2 = n\,[\mathrm{her}_{p+n}\,l\varkappa + (-1)^n\,\mathrm{her}_{p-n}\,l\varkappa]\,(\varkappa\,\mathrm{ber}_n'\,\varkappa - \mathrm{ber}_n\,\varkappa) - n\,[\mathrm{hei}_{p+n}\,l\varkappa + (-1)^n\,\mathrm{hei}_{p-n}\,l\varkappa] \times (\varkappa\,\mathrm{bei}_n'\,\varkappa - \mathrm{bei}_n\,\varkappa);\quad D_{np}^2 = 2\cdot\frac{(p+n-1)!}{(p-1)!\,(n-2)!}\cdot\frac{1}{l^{p+n}};$$

$$A_{np}^3 = [\mathrm{her}_{p+n}\,l\varkappa + (-1)^n\,\mathrm{her}_{p-n}\,l\varkappa]\,[(1-\nu)\,\mathrm{ber}_n'\,\varkappa - \nu\,\mathrm{bei}_n\,\varkappa] - [\mathrm{hei}_{p+n}\,l\varkappa + (-1)^n\,\mathrm{hei}_{p-n}\,l\varkappa]\,[(1-\nu)\,\mathrm{bei}_n'\,\varkappa + \nu\,\mathrm{ber}_n\,\varkappa];$$

$$B_{np}^3 = -[\mathrm{her}_{p+n}\,l\varkappa + (-1)^n\,\mathrm{her}_{p-n}\,l\varkappa]\,[(1-\nu)\,\mathrm{bei}_n'\,\varkappa + \nu\,\mathrm{ber}_n\,\varkappa] - [\mathrm{hei}_{p+n}\,l\varkappa + (-1)^n\,\mathrm{hei}_{p-n}\,l\varkappa]\,[(1-\nu)\,\mathrm{ber}_n'\,\varkappa - \nu\,\mathrm{bei}_n\,\varkappa];$$

$$C_{np}^3 = 2\frac{1-\nu}{\varkappa^2}\cdot\frac{(p+n-1)!}{(p-1)!\,(n-2)!}\cdot\frac{1}{l^{p+n}};\quad A_{np}^4 = [\mathrm{her}_{p+n}\,l\varkappa + (-1)^n\,\mathrm{her}_{p-n}\,l\varkappa]\left[\mathrm{bei}_n'\,\varkappa + n^2\frac{1-\nu}{\varkappa^3}(\varkappa\,\mathrm{ber}_n'\,\varkappa - \mathrm{ber}_n\,\varkappa)\right] + [\mathrm{hei}_{p+n}\,l\varkappa + (-1)^n\,\mathrm{hei}_{p-n}\,l\varkappa]\left[\mathrm{ber}_n'\,\varkappa - n^2\frac{1-\nu}{\varkappa^3}(\varkappa\,\mathrm{bei}_n'\,\varkappa - \mathrm{bei}_n\,\varkappa)\right];$$

$$B_{np}^4 = [\mathrm{her}_{p+n}\,l\varkappa + (-1)^n\,\mathrm{her}_{p-n}\,l\varkappa]\left[\mathrm{ber}_n'\,\varkappa - n^2\frac{1-\nu}{\varkappa^3} \times (\varkappa\,\mathrm{bei}_n'\,\varkappa - \mathrm{bei}_n\,\varkappa)\right] - [\mathrm{hei}_{p+n}\,l\varkappa + (-1)^n\,\mathrm{hei}_{p-n}\,l\varkappa]\left[\mathrm{bei}_n'\,\varkappa + n^2\frac{1-\nu}{\varkappa^3}(\varkappa\,\mathrm{ber}_n'\,\varkappa - \mathrm{ber}_n\,\varkappa)\right];\quad C_{np}^4 = 2\frac{1-\nu}{\varkappa^3}\cdot\frac{n\,(p+n-1)!}{(p-1)!\,(n-2)!}\cdot\frac{1}{l^{p+n}}.$$

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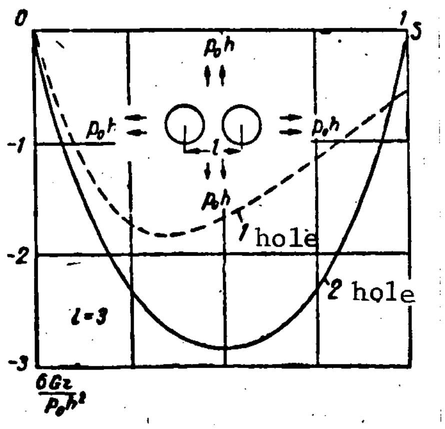

Figure 3

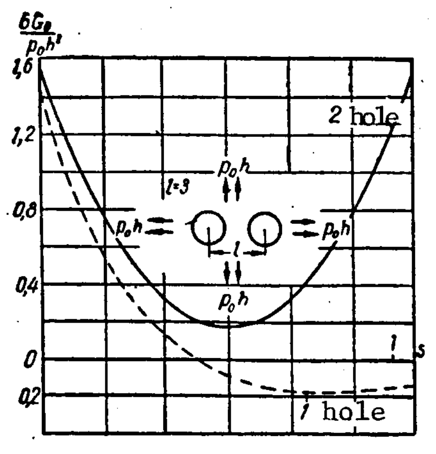

Figure 4

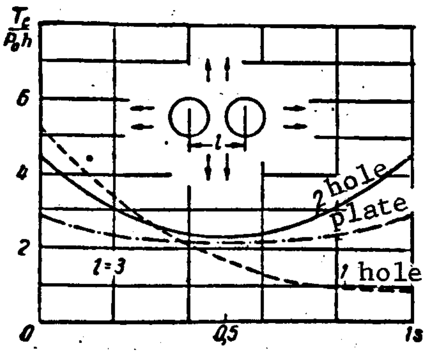

Figure 5

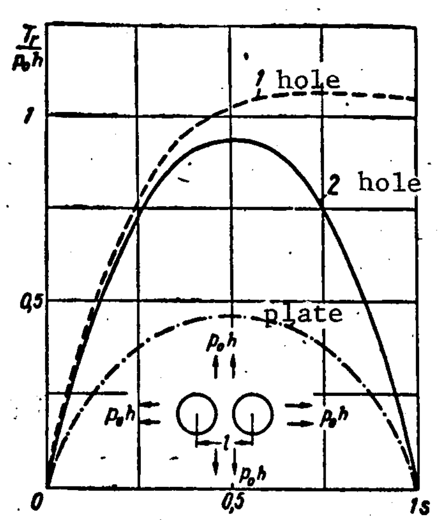

Figure 6

We should point out that we may designate $\frac{k(k+1)}{k!}$ by $\frac{1}{(k-2)!}$. /116

The second equation t = 2 vanishes in the system (11a and b) in the case n = 0, and the first and second equations coincide. In the case n = 1, the first and second equations (t = 1, 2) also coincide.

It is not possible to reduce the infinite system of algebraic equations (11a and b) to canonical form (Ref. 5). Therefore, it cannot be determined whether the system is regular. Let us proceed as follows. We shall replace the infinite system (11a and b) in the $j^{\underline{th}}$ approximation by the finite system

$$A_n^t a_n^{(j)} + B_n^t b_n^{(j)} + C_n^t c_n^{(j)} + D_n^t d_n^{(j)} + \varepsilon_n \sum_{p=0}^{j} (A_{np}^t a_p^{(j)} + B_{np}^t b_p^{(j)} + C_{np}^t c_p^{(j)} + D_{np}^t d_p^{(j)}) = -p_0 \frac{R}{E} \delta_n^0 (\delta_t^1 + \varkappa \delta_t^4); \quad t = 1, 2, 3, 4; \quad n = 0, 1, \ldots, j. \tag{12}$$

In the $j^{\underline{th}}$ approximation, we shall show that the boundary conditions are satisfied at the most characteristic point A (see Figure 2). Many authors have employed a similar procedure when studying the plane problem, and the problems of torsion and bending of rods in the case of multiply connected regions. In view of the symmetry, the boundary conditions for $S_{r_1\theta_1}$ are satisfied exactly at the point A, and we shall not examine the boundary conditions for $\tilde{Q}_{r_1}$, since $\tilde{Q}_{r_1}$ attentuates very rapidly (Ref. 8-10) and its value is insignificant. Thus, we shall determine whether the boundary conditions are satisfied for T_{r_1} and G_{r_1}. In order that these quantities be dimensionless, we shall compare $\frac{T_{r_1}}{p_0 h}$ and $6\frac{G_{r_1}}{p_0 h^2}$ (coefficients for the concentration of shearing and the maximum bending stresses) with unity (the principal stress state).

Let us investigate a shell with the radius R = 200 cm; r_0 = 10 cm; h = = 0.2 cm; ν = 0.3 for ℓ = 3 and ℓ = 4. For one hole, we have

$$\left.\frac{T_\theta}{p_0 h}\right|_A = 5{,}296;\quad 6\left.\frac{G_\theta}{p_0 h^2}\right|_A = 1{,}407. \tag{13}$$

In the plate $\left.\frac{T_\theta}{p_0 h}\right|_A$ = 2.000.

1. ℓ = 3. If we do not take into account the reciprocal effect, then we have

$$\left.\frac{T_r}{p_0 h}\right|_A = 0{,}049;\quad 6\left.\frac{G_r}{p_0 h^2}\right|_A = -0{,}589, \tag{14}$$

i.e., the maximum error in satisfying the boundary conditions is 59%. The results derived from solving the system (12) in the case j = 5 are given in the table. The maximum error for the stresses is 0.5% and for $6\frac{G_r}{p_0 h^2}$ - 4%.

2. ℓ = 4. In this case, it is sufficient to take j = 0 in (12) in order to find the solution with the same accuracy as in the preceding case. The results derived from the calculation for the points A, B, C and D (see Figure 2) are given in the table. The values of the corresponding coefficients for the plate are given in the denominator (Ref. 14). As may be seen from the table, $k_{max}|_A$ in a shell in the case of two holes for ℓ = 3 is /117

$$k_{2\max}\big|_A = \left.\frac{T_\theta}{p_0 h}\right|_A + 6\left.\frac{G_\theta}{p_0 h^2}\right|_A = 5{,}973,$$

in the case of one hole $k_{1max}|_A$ = 5.296 + 1.407 = 6.703. In the plate, $k_{2max}|_A$ = 2.89 and $k_{1max}|_A$ = 2.00. Figures 3-6 show the distribution $\frac{1}{p_0 h}T_r$, $\frac{1}{p_0 h}T_\theta$, $\frac{6}{p_0 h^2}G_r$ and $\frac{6}{p_0 h^2}G_\theta$ over the line connecting the hole centers. The following conclusions may be drawn from these figures and the table:

Points	l	$\frac{1}{p_0 h} T_r$	$\frac{1}{p_0 h} T_\theta$	$6 \frac{1}{p_0 h^2} G_r$	$6 \frac{1}{p_0 h^2} G_\theta$
A	3	$\frac{-0{,}005}{0{,}00}$	$\frac{4{,}445}{2{,}89}$	−0,040	1,528
	4	$\frac{0{,}000}{0{,}00}$	$\frac{5{,}280}{2{,}41}$	−0,040	1,416
B	3	0,567	3,247	−1,832	0,772
	4	0,923	2,298	−1,815	−0,568
C	3	0,853	2,496	−2,613	0,331
	4	1,079	1,040	−1,456	−0,753
D	3	$\frac{0{,}938}{0{,}46}$	$\frac{2{,}246}{2{,}16}$	−2,829	0,190
	4	$\frac{1{,}105}{0{,}62}$	$\frac{0{,}709}{1{,}53}$	−1,151	−0,265

1. The method presented makes it possible to study the stress distribution when the holes converge by controlling the accuracy with which the boundary conditions are satisfied.

2. When the holes converge (the case $\ell \geq 3$ which is investigated) there is a decrease in the stress concentration in the shell along the center lines, and T_θ decreases substantially. At the same time, there is an increase in the /119
stress concentration in the plate (we are making a comparision with one hole here).

We should point out that the results obtained pertain to the case of non-reinforced holes.

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STRESS STATE OF A SPHERICAL SHELL WEAKENED BY TWO CURVILINEAR HOLES /120

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N67-24516

The studies (Ref. 7-9) investigated the stress state in a spherical shell weakened by two circular holes. This problem was solved by the Schwarz method (Ref. 4). This article employs an approximate method for determining the stress state in a spherical shell weakened by two curvilinear holes.

Let us examine the equilibrium of a spherical shell weakened by two curvilinear holes. The stress state of this shell consists of the principal stress state and an additional stress state produced due to the holes (Ref. 8). We shall assume that the holes have such dimensions and are located at such a distance that the additional stress state may be described by the equations given by the theory of shallow shells (Ref. 2). The principal stress state describes the stress state in a non-weakened shell. The problem of determining the additional stress state may be reduced to integrating the equation

$$\nabla^2\nabla^2\Phi - i\varkappa^2\nabla^2\Phi = 0 \qquad (1)$$

under specific boundary conditions on the hole profiles and under conditions "at infinity" (Ref. 8).

Here we have

$$\Phi = w + i\lambda\varphi;\quad \varkappa^2 = \frac{\sqrt{12(1-\nu^2)}}{Rh}r_0^2;\quad \lambda = \frac{\sqrt{12(1-\nu^2)}}{Eh^2},$$

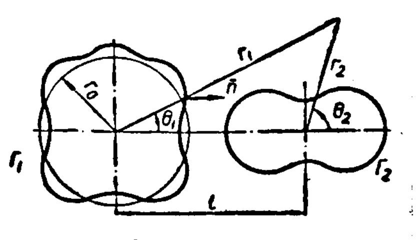
where r_0 is the parameter characterizing the absolute dimensions of the hole; R, h, E are the radius, thickness, and shell elasticity modulus, respectively.

The hole profiles are formed in such a way that the function /121

$$z_\kappa = \zeta_\kappa + \varepsilon\alpha_\kappa f_\kappa(\zeta_\kappa) \quad (z_\kappa = r_\kappa e^{i\theta_\kappa},\quad \zeta_\kappa = \rho_\kappa e^{i\gamma},\quad \varepsilon < 1) \qquad (2)$$

performs the conformal mapping of an infinite plane with a circular hole having unit raidus onto an infinite plane with a given hole. Here r_k are the dimensionless coordinates with respect to r_0 (see the figure).

We shall try to find the solution of the boundary value problem in the following form



$$\Phi = \sum_{j=0}^{\infty}\varepsilon^j[\Phi_{j1}(r_1,\theta_1) + \Phi_{j2}(r_2,\theta_2)] + \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\varepsilon^j[\Phi_{j12}^{(k)}(r_1,\theta_1) + \Phi_{j21}^{(k)}(r_2,\theta_2)]. \qquad (3)$$

Here $\Phi_{j1}(r_1, \theta_1)$ and $\Phi_{j2}(r_2, \theta_2)$ are the solutions, respectively, for each curvilinear hole (Ref. 3). $\Phi^{(k)}_{j12}(r_1, \theta_1)$ represents the deviation of the profile of the first hole from the function $\Phi^{(k-1)}_{j21}(r_2, \theta_2)$. $\Phi^{(k)}_{j21}(r_2, \theta_2)$ represents the deviation of the second hole profile from the function $\Phi^{(k-1)}_{j12}(r_1, \theta_1)$.

Thus, the Schwarz method is employed, but in each successive approximation the functions are represented in the form of a series with respect to the parameter characterizing the deviation of the hole form from a circular form (Ref. 5).

The solution of equation (1) which satisfies the conditions at infinity (Ref. 6), has the following form

$$\Phi(r, \theta) = iC \ln r + (C_0 + iD_0) H_0^{(1)}(\sqrt{-i}\varkappa r) + \\ + \sum_{n=1}^{\infty} [(A_n + iB_n) r^{-n} + (C_n + iD_n) H_n^{(1)}(\sqrt{-i}\varkappa r)] \cos n\theta, \quad (4)$$

where C, C_n, D_n, A_n, B_n are the constants; $H_n^{(1)}(\sqrt{-i}\kappa r) = \mathrm{her}_n \kappa r + i\,\mathrm{hei}_n \kappa r$ -- the Hankel functions of the first kind of the n^{th} order (Ref. 1). The constants for all of the functions may be determined according to the method given in (Ref. 11) from the boundary conditions, which have the following form in the case of free profiles on each hole:

$$T_n|_{r_0} = -\frac{pR}{2}; \quad S_{ns}|_{r_0} = 0; \quad G_n|_{r_0} = 0; \quad \tilde{Q}_n|_{r_0} = F(\gamma, \varepsilon), \quad (5)$$

where $F(\gamma, \varepsilon)$ is given in (Ref. 3).

With such a formulation, we may solve the problem for a wide class of hole forms by the appropriate selection of the mapping function $f(\zeta)$.

Let us examine a shell weakened by a circular and an elliptical hole, which is loaded by internal pressure. Only the intersection forces are in operation on the hole profiles.

In this case $f(\zeta) = \frac{1}{\zeta}$; $r_o = \frac{a + b}{2}$; $\varepsilon = \frac{a - b}{a + b}$, where a, b are the ellipse semiaxes. /122

Taking the fact into consideration that an elliptical hole differs very little from a circular hole, in (3) we may confine ourselves to only terms with ε in the first power, i.e., we may confine ourselves to the first approximation. Thus, the solution (3) will have the following form

$$\Phi = \Phi_{01}(r_1, \theta_1) + \Phi_{02}(r_2, \theta_2) + \varepsilon\Phi_{12}(r_2, \theta_2) + \\ + \sum_{k=1}^{\infty} [\Phi^{(k)}_{012}(r_1, \theta_1) + \Phi^{(k)}_{021}(r_2, \theta_2) + \varepsilon(\Phi^{(k)}_{112}(r_1, \theta_1) + \Phi^{(k)}_{121}(r_2, \theta_2))]. \quad (6)$$

The functions contained in (6) may be written as follows

$$\Phi_{01}(r_1, \theta_1) = (C^{(01)} + iD^{(01)}) H_0(\sqrt{-i}\varkappa r_1); \quad (7a)$$

$$\Phi_{02}(r_2, \theta_2) = (C^{(02)} + iD^{(02)}) H_0(\sqrt{-i}\varkappa r_2); \quad (7b)$$

$$\Phi_{12}(r_2, \theta_2) = (C_2^{(12)} + iD_2^{(12)}) H_2(\sqrt{-i}\varkappa r_2)\cos 2\theta_2 + + (A_2^{(12)} + iB_2^{(12)}) r_2^{-2}\cos 2\theta_2; \quad (7c)$$

$$\Phi_{j12}^{(k)}(r_1, \theta_1) = iC_\kappa^{(j12)}\ln r_1 + \sum_{n=1}^{\infty}[(A_{\kappa n}^{(j12)} + iB_{\kappa n}^{(j12)}) r_1^{-n} + + (C_{\kappa n}^{(j12)} + iD_{\kappa n}^{(j12)}) H_n(\sqrt{-i}\varkappa r_1)]\cos n\theta_1; \quad (7d)$$

$$\Phi_{j21}^{(k)}(r_2, \theta_2) = iC_\kappa^{(j21)}\ln r_2 + \sum_{n=1}^{\infty}[(A_{\kappa n}^{(j21)} + iB_{\kappa n}^{(j21)}) r_2^{-n} + + (C_{\kappa n}^{(j21)} + iD_{\kappa n}^{(j21)}) H_n(\sqrt{-i}\varkappa r_2)]\cos n\theta_2. \quad (7e)$$

The function $\Phi_{01}(r_1, \theta_1)$ characterizing the stress state around one circular hole was obtained in (Ref. 12); $\Phi_{02}(r_2, \theta_2)$ and $\Phi_{12}(r_2, \theta_2)$ for one elliptical hole were given in (Ref. 10).

If we confine ourselves to the reciprocal influence of holes in the first approximation (k = 1), the functions $\Phi_{012}^{(1)}$ and $\Phi_{021}^{(1)}$ coincide, respectively, with the functions $\Phi_{12}^{(1)}$ and $\Phi_{21}^{(1)}$ (Ref. 8).

Thus, in the first approximation the problem may be reduced to determining the functions $\Phi_{112}^{(1)}(r_1, \theta_1)$ and $\Phi_{121}^{(1)}(r_2, \theta_2)$. The arbitrary constants included in the functions $\Phi_{112}^{(1)}$ and $\Phi_{121}^{(1)}$ may be determined from the following system of equations

$$\left(\frac{1}{r_1}\cdot\frac{\partial}{\partial r_1} + \frac{1}{r_1^2}\cdot\frac{\partial^2}{\partial\theta_1^2}\right)\operatorname{Im}(\Phi_{112}^{(1)} + \Phi_{12})|_{r_1^\circ} = 0;$$

$$-\frac{\partial^2}{\partial r_1\partial\theta_1}\cdot\frac{1}{r_1}\operatorname{Im}(\Phi_{112}^{(1)} + \Phi_{12})|_{r_1^\circ} = 0; \quad (8)$$

$$-\left[(1-\nu)\frac{\partial^2}{\partial r_1^2} + \nu\nabla^2\right]\operatorname{Re}(\Phi_{112}^{(1)} + \Phi_{12})|_{r_1^\circ} = 0; \quad (8a)$$

$$-\left(\frac{\partial}{\partial r_1}\nabla^2 + \frac{1-\nu}{r_1}\cdot\frac{\partial^3}{\partial r_1\partial\theta_1^2}\cdot\frac{1}{r_1}\right)\operatorname{Re}(\Phi_{112}^{(1)} + \Phi_{12})|_{r_1^\circ} = 0; \quad (8b)$$ /123

$$\left(\frac{1}{r_2}\cdot\frac{\partial}{\partial r_2} + \frac{1}{r_2^2}\cdot\frac{\partial^2}{\partial\theta_2^2}\right)\operatorname{Im}\Phi_{121}^{(1)}\Big|_{\substack{r_2=1\\ \theta_2=\gamma}} = \left[L_1^{(1)}\left(\frac{1}{r_2}\cdot\frac{\partial}{\partial r_2} + \frac{1}{r_2^2}\cdot\frac{\partial^2}{\partial\theta_2^2}\right) - - L_3^{(1)}\frac{\partial^2}{\partial r_2\partial\theta_2}\cdot\frac{1}{r_2}\right]\operatorname{Im}(\Phi_{01} + \Phi_{021}^{(1)})\Big|_{\substack{r_2=1\\ \theta_2=\gamma}}; \quad (8c)$$

$$-\frac{\partial^2}{\partial r_2\partial\theta_2}\cdot\frac{1}{r_2}\operatorname{Im}\Phi_{121}^{(1)}\Big|_{\substack{r_2=1\\ \theta_2=\gamma}} = -\left[-L_1^{(1)}\frac{\partial^2}{\partial r_2\partial\theta_2}\cdot\frac{1}{r_2} + + \frac{1}{r_2}L_3^{(1)}\left(2\frac{\partial^2}{\partial r_2^2} - \nabla^2 2\right)\right]\operatorname{Im}(\Phi_{01} + \Phi_{021}^{(1)})\Big|_{\substack{r_2=1\\ \theta_2=\gamma}}; \quad (9a)$$

$$-\left[(1-\nu)\frac{\partial^2}{\partial r_2^2} + \nu\nabla^2\right]\operatorname{Re}\Phi_{121}^{(1)}\Big|_{\substack{r_2=1\\ \theta_2=\gamma}} = \left\{L_1^{(1)}\left[(1-\nu)\frac{\partial^2}{\partial r_2^2} + \nu\nabla^2\right] + \right.$$

$$+(1-\nu)L_3^{(1)}\frac{\partial^2}{\partial r_2\partial\theta_2}\cdot\frac{1}{r_2}\Big\}\operatorname{Re}(\Phi_{01}+\Phi_{021}^{(1)})\Big|_{\substack{r_2=1\\ \theta_2=\gamma}};\tag{9b}$$

$$-\left(\frac{\partial}{\partial r_2}\nabla^2+\frac{1-\nu}{r_2}\cdot\frac{\partial^3}{\partial r_2\partial\theta_2^2}\cdot\frac{1}{r_2}\right)\operatorname{Re}\Phi_{121}^{(1)}\Big|_{\substack{r_2=1\\ \theta_2=\gamma}}=L_4^{(1)}\operatorname{Re}(\Phi_{01}+\Phi_{021}^{(1)})\Big|_{\substack{r_2=1\\ \theta_2=\gamma}}.\tag{9c}$$

The stress T_s on the profile Γ may be written in the following form

$$\begin{aligned}T_{s|\Gamma}=\frac{pR}{2}+\frac{1}{\lambda r_o^2}\sum_{j=0}^{\infty}\varepsilon^j\Big\{\frac{\partial^2}{\partial r_2^2}+\sum_{m=0}^{j-1}\Big[L_1^{(j-m)}\frac{\partial^2}{\partial r_2^2}+\\ +L_2^{(j-m)}\left(\frac{1}{r_2}\cdot\frac{\partial}{\partial r^2}+\frac{1}{r_2^2}\cdot\frac{\partial^2}{\partial\theta_2^2}-\frac{\partial^2}{\partial r_2^2}\right)+L_3^{(j-m)}\frac{\partial^2}{\partial r_2\partial\theta_2}\cdot\frac{1}{r_2}\Big]\Big\}\times\\ \times\Big\{\Phi_{j2}^{(0)}+\sum_{k=1}^{\infty}[\Phi_{j12}^{(k)}+\Phi_{j21}^{(k)}]\Big\}\Big|_{\substack{r_2=1\\ \theta_2=\gamma}}+\frac{1}{\lambda r_o^2}\cdot\frac{\partial^2}{\partial r_2^2}\Phi_{01}+\\ +\frac{1}{\lambda r_o^2}\sum_{j=1}^{\infty}\varepsilon^j\Big[L_1^{(j)}\frac{\partial^2}{\partial r_2^2}+L_2^{(j)}\left(\frac{1}{r^2}\cdot\frac{\partial}{\partial r_2}+\frac{1}{r_2^2}\cdot\frac{\partial^2}{\partial\theta_2^2}-\frac{\partial^2}{\partial r_2^2}\right)+\\ +L_3^{(j)}\frac{\partial^2}{\partial r_2\partial\theta_2}\cdot\frac{1}{r_2}\Big]\Phi_{01}\Big|_{\substack{r_2=1\\ \theta_2=\gamma}}.\end{aligned}\tag{10}$$

The operators $L_1^{(j)}$, $L_2^{(j)}$, $L_3^{(j)}$, $L_4^{(j)}$ are given in (Ref. 11).

Let us investigate the influence of an elliptical hole upon the stress state around a circular hole. In order to do this, in (6) it is sufficient to confine ourselves to the following terms

$$\Phi=\Phi_{01}+\Phi_{02}+\varepsilon\Phi_{12}+\Phi_{012}^{(1)}+\varepsilon\Phi_{112}^{(1)}.\tag{11}$$

With the exception of $\Phi_{112}^{(1)}$, all of the functions included in (11) were determined in (Ref. 8, 10, 12). The constants were determined from the system (8a) - (8c). In the case when R = 200 cm, ℓ = 3, $r_1 = r_o$ = 100 cm, ν = 0.3, h = 0.2, the stress value on the profile of a circular hole at the point /124
θ_1 = 0 (see the figure) has the following form

$$T_s\Big|_{\substack{\Gamma\\ \theta_1=0}}=\frac{pR}{2}\left(4{,}5811-\frac{a-b}{a+b}2{,}7691\right).\tag{12}$$

The second term in (12) characterizes the influence of ellipticity of the second hole on the stress state on the profile of a circular hole.

The table presents the coefficient of the stress concentration $K=\dfrac{T_s|_{\Gamma\,\theta=0}}{\frac{pR}{2}}$ for different values of the ellipse semiaxes. The lower line gives the error δ (in percents), which is assumed if we calculate the influence of an elliptical hole as the influence of a circular hole with the radius $r_o = \dfrac{a+b}{2}$.

ε	0,20	0,15	0,13	0,09	0,05	0,00	−0,05	−0,09	−0,13	−0,15	−0,20
a/b	1,50	1,40	1,30	1,20	1,10	1,00	$\frac{1}{1,10}$	$\frac{1}{1,20}$	$\frac{1}{1,30}$	$\frac{1}{1,40}$	$\frac{1}{1,50}$
K	4,03	4,16	4,22	4,34	4,44	4,58	4,72	4,83	4.94	4,99	5,13
δ	−12	−9	−8	−5	−3	0	+3	+5	+8	+9	+12

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FORMULATION OF STATIC BOUNDARY VALUE PROBLEMS FOR SHALLOW SHELLS FOR MULTIPLY CONNECTED REGIONS[1]

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N67-24517

This article investigates a multiply connected shell referred to an arbitrary orthogonal coordinate system α, β. The equations of statics for shallow shells given by V. Z. Vlasov (Ref. 2), which may be written in terms of the functions F and w, have the following form

$$\frac{1}{Eh}\Delta\Delta F - \Delta_k w = 0; \tag{1}$$

$$\Delta_k F + D\Delta\Delta w - Z = 0. \tag{2}$$

Here we have

$$\Delta = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\cdot\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\cdot\frac{\partial}{\partial\beta}\right)\right];$$

$$\Delta_k = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}k_2\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}k_1\frac{\partial}{\partial\beta}\right)\right].$$

The stress function F and its normal derivative $\frac{\partial F}{\partial n}$ are not given directly on the shell profile, and only the stresses related to F and $\frac{\partial F}{\partial n}$ by the following differential relationships are known:

$$N_1 = \frac{1}{B}\cdot\frac{\partial}{\partial\beta}\left(\frac{1}{B}\cdot\frac{\partial F}{\partial\beta}\right) + \frac{1}{A^2B}\cdot\frac{\partial B}{\partial\alpha}\cdot\frac{\partial F}{\partial\alpha}; \tag{3}$$

$$N_2 = \frac{1}{A}\cdot\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\cdot\frac{\partial F}{\partial\alpha}\right) + \frac{1}{AB^2}\cdot\frac{\partial A}{\partial\beta}\cdot\frac{\partial F}{\partial\beta}; \tag{4}$$

$$S = -\frac{1}{AB}\left(\frac{\partial^2 F}{\partial\alpha\cdot\partial\beta} - \frac{1}{B}\cdot\frac{\partial B}{\partial\alpha}\cdot\frac{\partial F}{\partial\beta} - \frac{1}{A}\cdot\frac{\partial A}{\partial\beta}\cdot\frac{\partial F}{\partial\alpha}\right). \tag{5}$$

Due to this fact, for a multiply connected region the boundary conditions with respect to F and $\frac{\partial F}{\partial n}$ include additional unknown values. /126

The studies (Ref. 3 and 4) investigated the problem of determining them for the case when the shell profile coincides with a coordinate line in a Cartesian or polar coordinate system.

Let us investigate the more general problem of formulating the boundary conditions for F and $\frac{\partial F}{\partial n}$ on the profile, which represents the coordinate line $\alpha = \alpha_0$ in a certain arbitrary orthogonal system of coordinates α, β. Another

[1] This work was performed in the Laboratory of Thin-Walled Three-Dimensional Objects of the Kiev Engineering Construction Institute, under the supervision of Professor D. V. Vaynberg.

method of deriving some of the results given in (Ref. 3-4) will be obtained as a special case.

Let us set the following condition on the profile $\alpha = \alpha_0$

$$\left.\begin{aligned} N_1 &= \psi_1(\beta) \\ S &= \psi_2(\beta) \end{aligned}\right\}. \tag{6}$$

We shall employ the following notation

$$\left.\begin{aligned} F_{\alpha=\alpha_0} &= f_1(\beta) \\ \frac{\partial F}{\partial \alpha}\bigg|_{\alpha=\alpha_0} &= f_2(\beta) \end{aligned}\right\}. \tag{7}$$

We find from relationships (3) and (4) that the functions f_1 and f_2 must satisfy the following system of ordinary differential equations:

$$\frac{1}{B}\cdot\frac{\partial}{\partial\beta}\left(\frac{1}{B}\cdot\frac{\partial f_1}{\partial\beta}\right)+\frac{1}{A^2B}\cdot\frac{\partial B}{\partial\alpha}\cdot f_2=\psi_1; \tag{8}$$

$$-\frac{1}{AB}\left(\frac{\partial f_2}{\partial\beta}-\frac{1}{B}\cdot\frac{\partial B}{\partial\alpha}\cdot\frac{\partial f_1}{\partial\beta}-\frac{1}{A}\cdot\frac{\partial A}{\partial\beta}\cdot f_2\right)=\psi_2 \tag{9}$$

or

$$\frac{\partial f_1}{\partial\beta}=\frac{\dfrac{\partial f_2}{\partial\beta}-\dfrac{1}{A}\cdot\dfrac{\partial A}{\partial\beta}f_2+AB\psi_2}{\dfrac{1}{B}\cdot\dfrac{\partial B}{\partial\alpha}}; \tag{10}$$

$$\frac{1}{B}\cdot\frac{\partial}{\partial\beta}\left[\frac{1}{\dfrac{\partial B}{\partial\alpha}}\left(\frac{\partial f_2}{\partial\beta}-\frac{1}{A}\cdot\frac{\partial A}{\partial\beta}f_2+AB\psi_2\right)\right]+\frac{1}{A^2B}\cdot\frac{\partial B}{\partial\alpha}\cdot f_2=\psi_1. \tag{11}$$

The general integral for the system of equations (10), (11) has the following form

$$f_2=f_{2(0)}+C_1f_{2(1)}+C_2f_{2(2)}; \tag{12}$$
$$f_1=f_{1(0)}+C_1f_{1(1)}+C_2f_{1(2)}+C_3, \tag{13}$$

where $f_{2(0)}$, $f_{z(1)}$ and $f_{2(2)}$ are determined from equation (11), and

$$f_{1(0)}=\int\left[\frac{B}{\dfrac{\partial B}{\partial\alpha}}\left(\frac{\partial f_{2(0)}}{\partial\beta}-\frac{1}{A}\cdot\frac{\partial A}{\partial\beta}f_{2(0)}+AB\psi_2\right)\right]\alpha\beta; \tag{14}$$

$$f_{1(1)}=\int\left[\frac{B}{\dfrac{\partial B}{\partial\alpha}}\left(\frac{\partial f_{2(1)}}{\partial\beta}-\frac{1}{A}\frac{\partial A}{\partial\beta}f_{2(1)}\right)\right]\alpha\beta; \tag{15}$$ /127

$$f_{1(2)}=\int\left[\frac{B}{\dfrac{\partial B}{\partial\alpha}}\left(\frac{\partial f_{2(2)}}{\partial\beta}-\frac{1}{A}\cdot\frac{\partial A}{\partial\beta}\cdot f_{2(2)}\right)\right]\alpha\beta. \tag{16}$$

Thus, the problem of determining the boundary values F and $\frac{\partial F}{\partial\alpha}$ on the profile $\alpha = \alpha_0$ may be reduced to determining the general integral of the differential equation (11).

If $\alpha = x$, $\beta = y$ are rectangular Cartesian coordinates, we obtain the following on the profile $x = x_0$

$$\left.\begin{aligned} F &= f_1 = f_{1\,(0)} + C_1 y + C_2 \\ \frac{\partial F}{\partial n} &= f_2 = f_{2\,(0)} + C_3 \end{aligned}\right\}. \tag{17}$$

We have the following in polar coordinates $\alpha = r$, $\beta = \theta$ on the profile $r = r_0$:

$$\left.\begin{aligned} F &= f_1 = f_{1\,(0)} + C_1 \cos\theta + C_2 \sin\theta + C_3 \\ \frac{\partial F}{\partial n} &= f_2 = f_{2\,(0)} + C_3 \end{aligned}\right\}. \tag{18}$$

In bipolar coordinates, equation (11) acquires the following form:

$$\frac{\partial^2 f_2}{\partial \beta^2} - \frac{3 \sin\beta}{\operatorname{ch}\alpha_0 + \cos\beta} \cdot \frac{\partial f_2}{\partial \beta} - \frac{2\cos\beta - \operatorname{ch}\alpha_0}{\operatorname{ch}\alpha_0 + \cos\beta} f_2 = 0. \tag{19}$$

In this case, we employed numerical integration. Thus, three unknown constants, which correspond to three powers of static indeterminancy (Ref. 1), are included in the composition of the boundary conditions for the system (10), (11) in the case of a doubly connected region.

We shall assume that the quantities C_1, C_2, C_3 are generalized forces. It follows from the condition of single-valued displacement of the shell that the generalized displacements corresponding to them equal zero.

Employing the Castigliano theorem, according to which the partial derivative of the potential energy with respect to one of the generalized forces equals the corresponding generalized displacement, we may write this condition as follows:

$$\frac{\partial V}{\partial C_1} = 0, \quad \frac{\partial V}{\partial C_2} = 0, \quad \frac{\partial V}{\partial C_3} = 0, \tag{20}$$

where

$$\begin{aligned} V = \frac{1}{2} \iint \Big(& \frac{N_1^2}{Eh} + \frac{N_2^2}{Eh} - 2\nu \frac{N_1 N_2}{Eh} + \frac{2(1+\nu)}{Eh} S^2 - D\varkappa_1^2 - \\ & - D\varkappa_2^2 - 2\nu D \varkappa_1 \varkappa_2 + D(1-\nu)\tau^2 \Big) AB \, d\alpha \, d\beta. \end{aligned} \tag{21}$$

We thus obtain the following system of linear equations /128

$$\left.\begin{aligned} \delta_{11}C_1 + \delta_{12}C_2 + \delta_{13}C_3 + \delta_{1p} = 0; \\ \delta_{21}C_1 + \delta_{22}C_2 + \delta_{23}C_3 + \delta_{2p} = 0; \\ \delta_{31}C_1 + \delta_{32}C_2 + \delta_{33}C_3 + \delta_{3p} = 0. \end{aligned}\right\} \tag{22}$$

In order to determine δ_{ik}($i = 1, 2, 3,$; $R = 1, 2, 3, p$) we must express the potential energy V explicitly in the form of the quadratic form of

C_1, C_2, C_3.

Let us introduce the following shell states into the examination:

$$\begin{aligned}&\text{state } C_1: C_1=1, \quad C_2=C_3=p=0;\\&\text{state } C_2: C_2=1, \quad C_1=C_3=p=0;\\&\text{state } C_3: C_3=1, \quad C_1=C_2=p=0;\\&\text{state } p: C_1=C_2=C_3=0, \quad p=p_0.\end{aligned}$$

The set of external forces acting upon the shell is designated symbolically by p. The boundary condtions are completely determined in each of the states investigated. We shall designate the quantities pertaining to each state by the superscripts in parenthesis. In view of the superposition principle, we have

$$\left.\begin{aligned}N_1&=N_1^{(p)}+N_1^{(1)}C_1+N_1^{(2)}C_2+N_1^{(3)}C_3;\\N_2&=N_2^{(p)}+N_2^{(1)}C_1+N_2^{(2)}C_2+N_2^{(3)}C_3;\\S&=S^{(p)}+S^{(1)}C_1+S^{(2)}C_2+S^{(3)}C_3;\\\varkappa_1&=\varkappa_1^{(p)}+\varkappa_1^{(1)}C_1+\varkappa_1^{(2)}C_2+\varkappa_1^{(3)}C_3;\\\varkappa_2&=\varkappa_2^{(p)}+\varkappa_2^{(1)}C_1+\varkappa_2^{(2)}C_2+\varkappa_2^{(3)}C_3;\\\tau&=\tau^{(p)}+\tau^{(1)}C_1+\tau^{(2)}C_2+\tau^{(3)}C_3.\end{aligned}\right\} \tag{23}$$

The derivatives of the stresses and deformations with respect to C_i have the following form

$$\begin{aligned}&\frac{\partial N_1}{\partial C_i}=N_1^{(i)}, \quad \frac{\partial N_2}{\partial C_i}=N_2^{(i)}, \quad \frac{\partial S}{\partial C_i}=S^{(i)};\\&\frac{\partial \varkappa_1}{\partial C_i}=\varkappa_1^{(i)}, \quad \frac{\partial \varkappa_2}{\partial C_i}=\varkappa_2^{(i)}, \quad \frac{\partial \tau}{\partial C_i}=\tau^{(i)}.\end{aligned} \tag{24}$$

Substituting (23) in (21) and taking (24) into account, we obtain

$$\begin{aligned}\delta_{ik}=\iint\Big\{&\frac{1}{Eh}[N_1^{(i)}N_1^{(k)}+N_2^{(i)}N_2^{(k)}-\nu N_1^{(i)}N_2^{(k)}-\nu N_1^{(k)}N_2^{(i)}+\\&+2(1+\nu)S^{(i)}S^{(k)}]-D[\varkappa_1^{(i)}\varkappa_1^{(k)}+\varkappa_2^{(i)}\varkappa_2^{(k)}+\nu\varkappa_1^{(i)}\varkappa_2^{(k)}+\\&+\nu\varkappa_1^{(k)}\varkappa_2^{(i)}+(1-\nu)\tau^{(i)}\tau^{(k)}]\Big\}AB\,d\alpha\,d\beta,\\&i=1,2,3;\quad k=1,2,3,p.\end{aligned} \tag{25}$$

In the case of rectangular Cartesian coordinates, we may transform the double integral (25) into a contour integral according to the Green formula: /129

$$\begin{aligned}\delta_{ik}=\frac{1}{Eh}\oint\Big[&-\left(\frac{\partial F_i}{\partial y}\cdot\frac{\partial^2 F_k}{\partial y^2}+\frac{\partial F_i}{\partial x}\cdot\frac{\partial^2 F_k}{\partial x\partial y}\right)+F_i\left(\frac{\partial^3 F_k}{\partial y^3}+\frac{\partial^3 F_k}{\partial x^2\partial y}\right)+\\&+\nu\left(\frac{\partial F_i}{\partial y}\cdot\frac{\partial^2 F_k}{\partial x^2}-\frac{\partial F_i}{\partial x}\cdot\frac{\partial^2 F_k}{\partial x\partial y}\right)\Big]dx+\Big[\frac{\partial F_i}{\partial x}\cdot\frac{\partial^2 F_k}{\partial x^2}+\frac{\partial F_i}{\partial y}\cdot\frac{\partial^2 F_k}{\partial x\partial y}-\\&-F_i\left(\frac{\partial^3 F_k}{\partial x^3}+\frac{\partial^3 F_k}{\partial x\partial y^2}\right)+\nu\left(\frac{\partial F_i}{\partial y}\cdot\frac{\partial^2 F_k}{\partial x\partial y}-\frac{\partial F_i}{\partial x}\cdot\frac{\partial^2 F_k}{\partial y^2}\right)\Big]dy+\end{aligned} \tag{26}$$

$$
\begin{aligned}
&+D\oint\Big[-\Big(\frac{\partial w_i}{\partial y}\cdot\frac{\partial^2 w_k}{\partial y^2}+\frac{\partial w_i}{\partial x}\cdot\frac{\partial^2 w_k}{\partial x\partial y}\Big)+w_i\Big(\frac{\partial^3 w_k}{\partial y^3}+\frac{\partial^3 w_k}{\partial x^2\partial y}\Big)-\\
&-\nu\Big(\frac{\partial w_i}{\partial y}\cdot\frac{\partial^2 w_k}{\partial x^2}-\frac{\partial w_i}{\partial x}\cdot\frac{\partial^2 w_k}{\partial x\partial y}\Big)\Big]dx+\Big[\frac{\partial w_i}{\partial x}\cdot\frac{\partial^2 w_k}{\partial x^2}+\frac{\partial w_i}{\partial y}\cdot\frac{\partial^2 w_k}{\partial x\partial y}-\\
&-w_i\Big(\frac{\partial^3 w_k}{\partial x^3}+\frac{\partial^3 w_k}{\partial x\partial y^2}\Big)-\nu\Big(\frac{\partial w_i}{\partial y}\cdot\frac{\partial^2 w_k}{\partial x\partial y}-\frac{\partial w_i}{\partial x}\cdot\frac{\partial^2 w_k}{\partial y^2}\Big)\Big]dy+\\
&+D\iint\Delta\Delta w_i w_k\,dx\,dy.
\end{aligned}
\tag{26}
$$

In each of the smooth sections of profile, we may employ integration by parts for several terms in formula (26). Then (26) acquires the following form

$$
\begin{aligned}
\delta_{ik}=\frac{1}{Eh}\oint\Big[&-\frac{\partial F_i}{\partial y}\cdot\frac{\partial^2 F_k}{\partial y^2}+F_i\Big(\frac{\partial^3 F_k}{\partial y^3}+2\frac{\partial^3 F_k}{\partial x^2\partial y}\Big)+\nu\Big(\frac{\partial F_i}{\partial y}\cdot\frac{\partial^2 F_k}{\partial x^2}+\\
&+F_i\frac{\partial^3 F_k}{\partial x^2\partial y}\Big)\Big]dx+\Big[\frac{\partial F_i}{\partial x}\cdot\frac{\partial F_k}{\partial x^2}-F_i\Big(\frac{\partial^3 F_k}{\partial x^3}+2\frac{\partial^3 F_k}{\partial x\partial y^2}\Big)-\\
&-\nu\Big(\frac{\partial F_i}{\partial x}\cdot\frac{\partial^2 F_k}{\partial y^2}+F_i\frac{\partial^3 F_k}{\partial x\partial y^2}\Big)\Big]dy+\sum\frac{2(1+\nu)}{Eh}F_i\frac{\partial^2 F_k}{\partial x\partial y}+\\
+D\oint\Big[&-\frac{\partial w_i}{\partial y}\cdot\frac{\partial^2 w_k}{\partial y^2}+w_i\Big(\frac{\partial^3 w_k}{\partial y^3}+2\frac{\partial^3 w_k}{\partial x^2\partial y}\Big)-\nu\Big(\frac{\partial w_i}{\partial y}\cdot\frac{\partial^2 w_k}{\partial x^2}+w_i\frac{\partial^3 w_k}{\partial y^2 dx}\Big)\Big]dx+\\
+\Big[&\frac{\partial w_i}{\partial x}\cdot\frac{\partial^2 w_k}{\partial x^2}-w_i\Big(\frac{\partial^3 w_k}{\partial x^3}+2\frac{\partial^3 w_k}{\partial x\partial y^2}\Big)+\nu\Big(\frac{\partial w_i}{\partial x}\cdot\frac{\partial^2 w_k}{\partial y^2}+w_i\frac{\partial^3 w_k}{\partial y^2 dx}\Big)\Big]dy+\\
&+\sum D2(1-\nu)\,w_i\frac{\partial^2 w_k}{\partial x\partial y}+D\iint\Delta\Delta w_i w_k\,dx\,dy.
\end{aligned}
\tag{27}
$$

The sum is extended to the corner points of the profile here.

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CRITICAL LOAD FOR REGIONS WEAKENED BY HOLES WITH CRACKS /130

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This article investigates the problem of determining the critical load for a thin, infinite plate weakened by an arbitrary elliptical hole with two equal macroscopic equilibrium cracks having the length ℓ, in the case of uniaxial tension of the plate (Figure 1).

The profile L of the hole with cracks is free from external loads. It is assumed that the plate material will be elastic up to fracture.

The boundary conditions (Ref. 5) for the stress functions $\phi(\zeta)$, $\psi(\zeta)$ have the following form in the case under consideration

$$\varphi(\sigma)+\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)}+\overline{\psi(\sigma)}=0; \tag{1}$$

$$\overline{\varphi(\sigma)}+\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\varphi'(\sigma)+\psi(\sigma)=0. \tag{2}$$

Here $\omega(\sigma)$ is the boundary value of the function

$$\omega(\zeta)=R\left[\frac{L(1+m)}{2}\left(\zeta+\frac{1}{\zeta}\right)+(1-m)\sqrt{\frac{L^2}{4}\left(\zeta+\frac{1}{\zeta}\right)^2-1}\right], \tag{3}$$

where $m=\frac{a-b}{a+b}$; $1<L<\infty$; R -- real parameter.

The function $\omega(\zeta)$, (3), conformally maps the exterior of the ellipse with semiaxes a, b and with two cuts (cracks) along the transverse axis in the z-plane (Figure 1) onto the exterior of the unit circle in the ζ plane. We shall investigate one of the holomorphic branches $\omega(\zeta)$ for which the second component in (3) has a positive value for $\zeta = 1$.

The mapping function $\omega(\zeta)$ (3) is an irrational function and has singularities on the unit circle γ which correspond to the angular points of the profile L. Therefore, just as in (Ref. 3, 7) we shall approximate the function $\omega(\zeta)$ (3) by the function /131

$$\omega_N(\zeta)=R_1\left[\zeta+\sum_{n=1}^{N}c_n\zeta^{1-2n}\right], \tag{4}$$

so that the coefficients c_n satisfy the condition

$$\omega_N'(\zeta)=R_1(1-\zeta^{-2})Q_N(\zeta), \tag{5}$$

where Q_N is a polynomial all of whose roots lie within the unit circle, R_1, c_n are the real parameters.

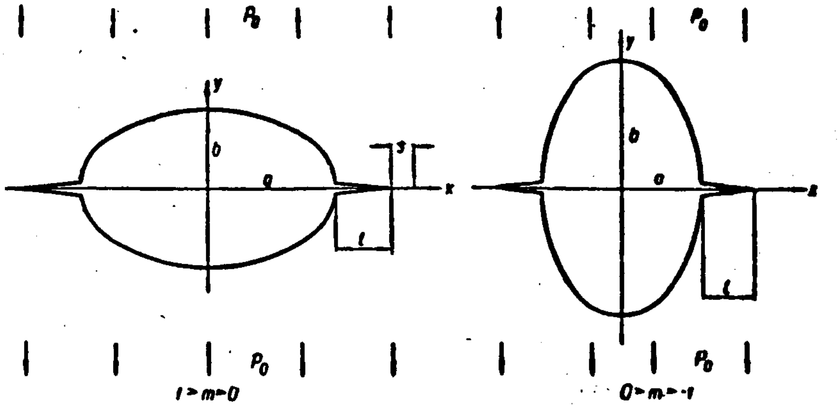

Figure 1

Due to this approximation, the cuspoidal points at the crack ends are retained on the new profile L'. Only the unstressed convex corners are curved at the points where the crack edges intersect the hole profile[1].

The stress functions which correspond to $\omega_N(\zeta)$ will be designated by $\phi_N(\zeta)$, $\psi_N(\zeta)$. Since the singularity of the solution, which arises due to the corner cuspidal points, can be completely obtained from the function $\phi_N(\zeta)$ (Ref. 2) ($\psi_N(\zeta)$ has a pole of the first order at the points $\zeta = \pm 1$) and does not influence the other stress function $\phi_N(\zeta)$, we shall obtain the function $\phi_N(\zeta)$ in the following form (Ref. 3)

$$\varphi_N(\zeta) = R_1 p_0 \left[\frac{\zeta}{4} + \sum_{n=1}^{N} \alpha_n \zeta^{1-2n}\right], \tag{6}$$

where α_n are the real coefficients; p_0 -- intensity of the tensile stresses.

The boundary condition (2) for the functions $\phi_N(\zeta)$, $\psi_N(\zeta)$ may be re- /132
written in the following form

$$\omega_N'(\sigma)\,\psi_N(\sigma) = -\omega_N'(\sigma)\,\varphi_N\left(\frac{1}{\sigma}\right) - \omega_N\left(\frac{1}{\sigma}\right)\varphi_N'(\sigma). \tag{7}$$

Comparing the coefficients in the expansion of the right and left sides of (7) for identical powers of σ, we find that the coefficients α_n must satisfy the following system of algebraic equations (Ref. 3):

$$\alpha_p + \sum_{n=1}^{N-p} \alpha_{n+p} c_n (1-2n) + \sum^{N-p} c_{n+p} \alpha_n (1-2n) + \tag{8}$$

[1] The function $\omega_N(\zeta)$ maps the exterior of a certain new profile L' onto $|\zeta|>1$.

$$+\frac{c_p}{4}=\begin{cases}0, & p>1;\\ -\frac{1}{2}, & p=1.\end{cases}$$
$$(p=1,\ 2,\ 3,\ \ldots,\ N). \tag{8}$$

Multiplying both parts of (7) by $\frac{1}{2\pi i}\cdot\frac{1}{\sigma-\zeta}$ and integrating over γ, we obtain

$$\omega_N'(\zeta)\,\psi_N(\zeta)=-\,\omega_N'(\zeta)\,\varphi_N\left(\frac{1}{\zeta}\right)-\omega_N\left(\frac{1}{\zeta}\right)\varphi_N'(\zeta). \tag{9}$$

Just as in (Ref. 3), in the case $N\to\infty$ the functions $\omega_N(\zeta)$, $\omega_N'(\zeta)$, $\phi_N(\zeta)$, $\phi_N'(\zeta)$, $\omega_N'(\zeta)\psi_N(\zeta)$ for real ζ outside of γ strive to $\omega(\zeta)$, $\omega'(\zeta)$, $\phi(\zeta)$, $\phi'(\zeta)$, $\omega'(\zeta)\psi(\zeta)$. Therefore, passing to the limit in the case $N\to\infty$ in the right and left sides of (9), we obtain the following for real z in the crack extensions

$$\psi(\zeta)=-\varphi\left(\frac{1}{\zeta}\right)-\frac{\omega(\zeta)}{\omega'(\zeta)}\varphi'(\zeta). \tag{10}$$

The stress components may be determined from the following relationships (Ref. 5)

$$X_x+Y_y=4\,\mathrm{Re}\{\Phi(\zeta)\}; \tag{11}$$
$$Y_y-X_x+2iX_y=2\left\{\overline{\frac{\omega(\zeta)}{\omega'(\zeta)}}\Phi'(\zeta)+\Psi(\zeta)\right\}. \tag{12}$$

In the case of uniaxial tension of the plate along the y-axis, the critical stresses for each of the cracks will be the same. Therefore, we may perform all the subsequent calculations for a crack which lies along the positive portion of the x axis. We may find the stress distribution $Y_y(x,\ 0)$ around the end of the crack under consideration from formulas (10) - (12):

$$Y_y(x,\ 0)=\frac{p_0}{2}\left\{\tilde{\varphi}'(\zeta)+\frac{1}{\zeta^2}\tilde{\varphi}'\left(\frac{1}{\zeta}\right)\right\}\sqrt{\frac{a+l}{2s}\cdot\frac{1-\frac{1}{(1+l_0)^4}}{1-\frac{m^2}{(1+l_0)^4}}}, \tag{13}$$

where /133

$$\tilde{\varphi}'(\zeta)=\frac{\varphi'(\zeta)}{R_1p_0};$$
$$1+l_0=\frac{1}{a+b}\left[(a+l)+\sqrt{(a+l)^2-(a^2-b^2)}\right],$$

where s is a small distance along the x axis from the point under consideration to the crack end (see Figure 1).

The tensile stresses will be critical, if the condition of G.I. Barenblatt is satisfied (Ref. 1)

$$Y_y(x,\ 0) = \frac{K_1}{\pi \sqrt{s}} + O(1), \tag{14}$$

where K_1 is the cohesion modulus.

Substituting (13) in (14), we obtain the expression for determining the critical stress:

$$p_{cr} = \frac{K_1}{\pi \tilde{\varphi}'(1)} \sqrt{\frac{2}{a+l} \cdot \frac{1 - \frac{m^2}{(1+l_0)^4}}{1 - \frac{1}{(1+l_0)^4}}}. \tag{15}$$

Since $\tilde{\phi}_N(1)$ is close to $\tilde{\phi}'(1)$ for large N, -- replacing $\tilde{\phi}'(1)$ in (15) by $\tilde{\phi}'_N(1)$ -- we obtain the approximate expression for determining the critical load[1]

$$p_{cr} = \frac{K_1}{\pi \tilde{\varphi}'_N(1)} \sqrt{\frac{2}{a+l} \frac{1 - \frac{m^2}{(1+l_0)^4}}{1 - \frac{1}{(1+l_0)^4}}}. \tag{16}$$

Determining the coefficients α_n from (8) and calculating the quantity (1) for different values of the parameters m and $\lambda = \frac{\ell}{R}$, from (16) we find the critical load which is necessary for the initial development of cracks. The calculations were performed on a BESM-2M computer for $1 \geq m \geq -1$. Up to 34 terms of the mapping function (4) were thus taken into account. Figure 2 presents curves showing the dependence of $\tilde{p}_{cr} = \frac{\pi p_{cr}}{K_1} \sqrt{R}$ on $\lambda = \frac{\ell}{R}$ for different ellipses. The form of the ellipse was changed by changing the parameter m.

In the case m = 1 (the insulated rectilinear crack) $\tilde{\phi}'(1) = 1$, and from (15) we obtain the well known formula for determining the critical load (Ref. 1)

$$p_{cr} = \frac{K_1}{\pi} \sqrt{\frac{2}{a+l}}. \tag{17}$$

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1 For ommidirectional tension of a plate, the critical load may also be determined from expression (16). However. the coefficients α_n may be determined from another system of equations given in (Ref. 3).

When m = -1, the ellipse changes into a vertical slit[1], and we obtain the formula p_{cr} for a cross-shaped crack from (16): /134

$$p_{cr} = \frac{K_1}{\pi \tilde{\varphi}'_N(1)} \sqrt{\frac{2}{l}}. \tag{18}$$

If we set $\tilde{\phi}'_N(1) = 1$, then formula (18) coincides with the well known formula for an insulated crack (Ref. 1). Actually, it may be seen from the graphs given in Figure 2 that the critical loads of an isolated and cross-shaped crack differ to an insignificant extent. Consequently, a vertical slit has a slight influence upon the development of a horizontal crack.

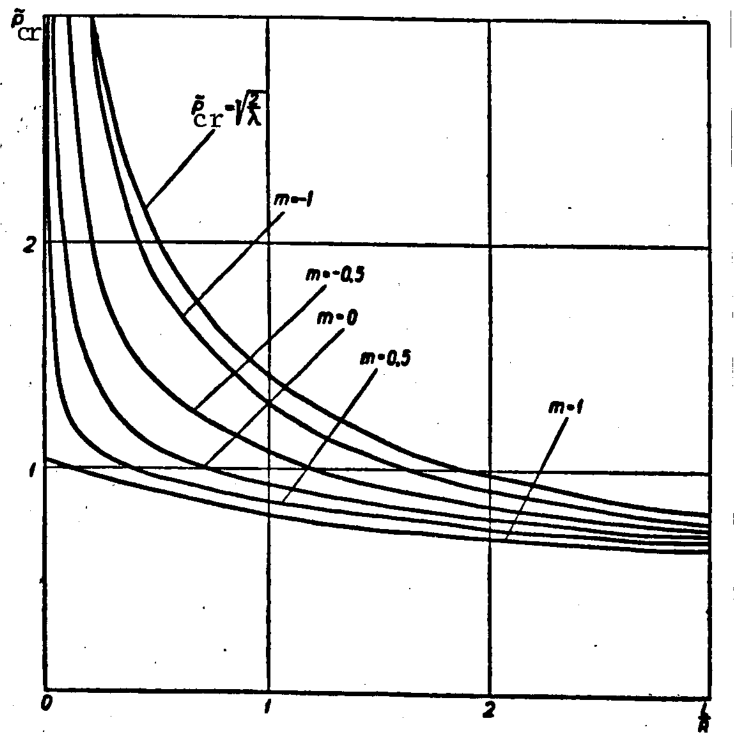

Figure 2

In the limiting case, for $\ell \to 0$ the values $\tilde{\phi}'_N(1)$ for different $0 \geq m \geq$ ≥ 1 are given in Tables 1 and 2, respectively, for omnidirectional and uniaxial plate tension. It may be seen from Tables 1 and 2 that an increase in the mapping function terms has only a slight influence upon $\tilde{\phi}'_N(1)$.

As follows from the statements given above, /135

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1 It is assumed that the edges of the vertical slit do not osculate.

TABLE 1

N	m						
	0	0,1	0,2	0,3	0,4	0,5	0,6
24	1,0000	1,0997	1,1992	1,2983	1,3965	1,4928	1,5845
34	1,0000	1,0999	1,1996	1,2992	1,3984	1,4968	1,5931

TABLE 2

N	m						
	0	0,1	0,2	0,3	0,4	0,5	0,6
24	1,4990	1,5485	1,5977	1,6463	1,6938	1,7392	1,7793
34	1,4995	1,5493	1,5989	1,6483	1,6972	1,7452	1,7908

$$\frac{p^*_{cr}}{p^{**}_{cr}} = \frac{[\tilde{\varphi}'_N(1)]^*}{[\tilde{\varphi}'_N(1)]^{**}},$$

where p^*_{cr}, $[\tilde{\phi}'_N(1)]^*$ are the values of these quantities in the case of omnidirectional tension, and p^{**}_{cr}, $[\tilde{\phi}'_N(1)]^{**}$ are the values of the same quantities in the case of uniaxial tension. Computing $\frac{[\tilde{\phi}'_N(1)]^*}{[\tilde{\phi}'_N(1)]^{**}}$ according to Tables 1 and 2, we find that this quantity is close to $\frac{k^*}{k^{**}}$, where k^* and k^{**} are the stress concentration coefficients in the elliptical hole apex under consideration without cracks for omnidirectional and uniaxial plate tension, respectively. Thus, $\frac{p^*_{cr}}{p^{**}_{cr}} \approx \frac{k^*}{k^{**}}$. This points to the fact that the local stress field around the hole greatly influences the development of small cracks. With an increase in the cracks, $\frac{p^*_{cr}}{p^{**}_{cr}} \to 1$, which indicates that the hole has a smaller influence upon the development of comparatively large cracks.

In the case $m = 0$ (circular hole), the curve giving the dependence of $\bar{p}_{cr}$ on $\frac{\ell}{R}$ (see Figure 2) is very similar to an analogous curve obtained in (Ref. 7) by the energy method of Griffith. /136

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TEMPERATURE STRESSES IN THIN PLATES WITH REINFORCED EDGES /137

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Heat exchange condition on the edge of thin plates reinforced by a thin rod. Let the edge of a thin plate having the thickness 2δ be reinforced by a ring made of another material having the same thickness and width $2\delta_k$ (Figure 1). Let us assume that the heat exchange between the system, which has the temperature t^0 at the initial moment of time τ, and the surrounding medium takes place in accordance with the Newton law, and let us also assume that there is an ideal thermal contact between the ring and the plate. We then have the following equation (Ref. 1) to determine the temperature field:

For a plate

$$\Delta t - \varkappa^2 (t - t_c) = \frac{c}{\lambda} \cdot \frac{\partial t}{\partial \tau}; \tag{1}$$

For a ring

$$\Delta t_\kappa - \varkappa_k^2 (t_\kappa - t_c) = \frac{c_\kappa}{\lambda_\kappa} \cdot \frac{\partial t_\kappa}{\partial \tau}; \tag{2}$$

The boundary conditions

$$\begin{gathered} \lambda \frac{\partial t}{\partial n} = \lambda_\kappa \frac{\partial t_\kappa}{\partial n},\ t = t_\kappa \text{ on } L;\ \lambda_\kappa \frac{\partial t_\kappa}{\partial n} = \alpha_c (t_\kappa - t_c) \text{ on } L_c, \\ \lambda \frac{\partial t}{\partial n'} = -\alpha_c' (t - t_c) \text{ on } L_c'; \end{gathered} \tag{3}$$

The initial condition

$$t = t_\kappa = t^0 \text{ for } \tau = 0, \tag{4}$$

where Δ is the Laplace operator; $\kappa^2 = \frac{\alpha}{\lambda\delta}$; $\kappa_k^2 = \frac{\alpha_k}{\lambda_k\delta}$; λ, λ_k -- thermal conductivity of the plate and the ring; c, c_k -- their specific heats; α, α_k -- heat transfer coefficients from the lateral surfaces $z = \pm\delta$ of the plate and the ring, respectively; α_c, α_c' -- heat transfer coefficients with L_c and L'_c; t, t_k, t_c -- temperature of the plate, ring, and medium; $\bar{n}'$ -- normal to L_c. /138

Assuming that the ring width $2\delta_k$ is of the same order of magnitude as the thickness 2δ, we shall regard it as a thin rod. Let us formulate the condition which the plate temperature must satisfy on the reinforced edge. We shall assume that the rod axis coincides with the plate profile. For this purpose, we shall relate the rod to the coordinates (s, n) and, writing (2) in these coordinates, we shall disregard the quantities kn (k -- curvature of the profile L_k) as compared with unity. As a result we obtain

$$p^2 t_\kappa + \frac{\partial^2 t_\kappa}{\partial n^2} = -\varkappa_\kappa^2 t_c, \tag{5}$$

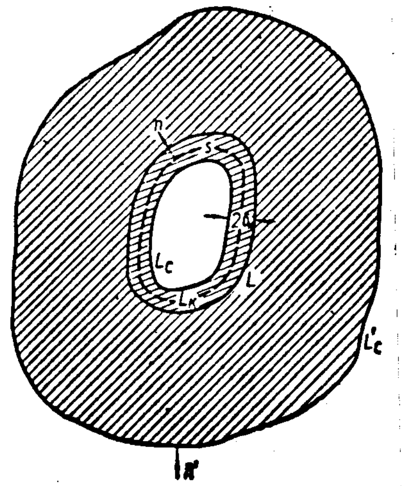

Figure 1

where

$$p^2 = \frac{\partial^2}{\partial s^2} - \varkappa_k^2 - \frac{c_k}{\lambda_k} \cdot \frac{\partial}{\partial \tau};$$ ds is

an element of the arc L_k. Let us introduce the integral characteristics;

$$T_k = \frac{1}{2\delta_k} \int_{-\delta_k}^{\delta_k} t_k dn; \quad T_k^* = \frac{3}{2\delta_k^2} \int_{-\delta_k}^{\delta_k} n t_k dn. \tag{6}$$

Averaging (5) in accordance with (6), we have

$$\begin{aligned} &p^2 T_k + \frac{1}{2\delta_k}\left[\left(\frac{\partial t_k}{\partial n}\right)^+ - \left(\frac{\partial t_k}{\partial n}\right)^-\right] = -\varkappa_k^2 T_c; \\ &p^2 T_k^* + \frac{3}{2\delta_k}\left[\left(\frac{\partial t_k}{\partial n}\right)^+ + \left(\frac{\partial t_k}{\partial n}\right)^-\right] - \frac{3(t_k^+ - t_k^-)}{2\delta_k^2} = -\varkappa_k^2 T_c^*, \end{aligned} \tag{7}$$

where

$$T_c = \frac{1}{2\delta_k} \int_{-\delta_k}^{\delta_k} t_c dn; \quad T_c^* = \frac{3}{2\delta_k^2} \int_{-\delta_k}^{\delta_k} n t_c dn,$$

where the indices + and - indicate the values of the functions in the case $n = \pm \delta_k$, respectively. /139

Let us determine T_k and T_k^* in terms of the boundary values of the temperature t_k in the case $n = \pm \delta_k$. For this purpose, we shall write the solution of equation (5) in the following form:

$$\begin{aligned} t_k = \frac{1}{2}\Bigg[&\frac{\cos pn}{\cos p\delta_k}(t_k^+ + t_k^-) + \frac{\sin pn}{\sin p\delta_k}(t_k^+ - t_k^-)\Bigg] + \\ &+ \frac{\varkappa_k^2}{p}\Bigg[\int_{\delta_k}^{n} \sin p(n_0 - n)\, t_c dn_0 + \\ &+ \frac{\sin p(\delta_k - n)}{\sin 2p\delta_k} \int_{-\delta_k}^{\delta_k} \sin p(n_0 + \delta_k)\, t_c dn_0\Bigg]. \end{aligned} \tag{8}$$

Substituting (8) in (6), we obtain

$$\begin{aligned} T_k &= \frac{\operatorname{tg} p\delta_k}{2p\delta_k}(t_k^+ + t_k^-) + \frac{\varkappa_k^2}{p^2}\left(\frac{1}{2\delta_k \cos p\delta_k}\int_{-\delta_k}^{\delta_k} \cos pn t_c dn - T_c^*\right); \\ T_k^* &= \frac{3}{2p^2\delta_k^2}(1 - p\delta_k \operatorname{ctg} p\delta_k)(t_k^+ - t_k^-) + \\ &+ \frac{\varkappa_k^2}{p^2}\left(\frac{3}{2\delta_k \sin p\delta_k}\int_{-\delta_k}^{\delta_k} \sin pn t_c dn - T_c^*\right). \end{aligned} \tag{9}$$

Instead of T_k and T_k^*, if we substitute their values (9) in (8) and if we take condition (3) into account, we eliminate t_k^- from the system of equations obtained. Then, setting

$$\lambda_\kappa^* = 2\lambda_\kappa\delta_\kappa;\ c_\kappa^* = 2c_\kappa\delta_\kappa;\ \alpha_\kappa^* = 2\alpha_\kappa\frac{\delta_\kappa}{\delta},$$

in the equation obtained we pass to the limit for $\delta_k \to 0$, thus retaining λ_k^*, C_k^*, α_k^*, r_k^* as constants. As a result, we obtain the following condition of heat exchange on the edge of a plate L_k which is reinforced by a thin rod;

$$\left[\lambda_\kappa^*\frac{\partial^2}{\partial s^2} + (1 + \alpha_c r_\kappa^*)\lambda\frac{\partial}{\partial n} - c_\kappa^*\frac{\partial}{\partial\tau}\right]t = (\alpha_c + \alpha_\kappa^*)(t - t_c), \tag{10}$$

which is characterized by four thermophysical parameters of the rod; heat resistance r_k^*, thermal conductivity λ_k^*, specific heat C_k^*, and heat transfer α_k^*. It may readily be seen that the Newton heat exchange condition for an unreinforced plate follows from (10) for zero values of all four thermal physical parameters of the rod. /140

Setting $r_c = \frac{1}{a_c} = 0$ in (10), we obtain the following condition

$$\lambda\frac{\partial t}{\partial n} = \frac{t - t_c}{r_\kappa^*}, \tag{11}$$

which coincides with the Newton condition for an unreinforced plate, in which the thermal resistance of the rod r_k^* plays the role of the heat exchange resistance r_c on the plate surface. In the case $r_k^* = 0$ we obtain the well-known condition of the first type from (11).

Temperature field and temperature stresses in an infinite plate with a circular reinforced edge. Let us determine the temperature field and the temperature stresses in an infinite plate which is free from external load and which has a circular reinforced edge. Assuming that the average temperature is only a function of time, we shall have the following, instead of (1) and (10):

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial t}{\partial r} - \mathrm{Bi}\,(t - t_c) = \frac{\partial t}{\partial \mathrm{Fo}}; \tag{12}$$

$$\frac{\partial t}{\partial r} - \mathrm{Bi}_0\,(t - t_c) = \gamma\frac{\partial t}{\partial \mathrm{Fo}} \text{ for } r = R, \tag{13}$$

where

$$\mathrm{Bi} = \frac{\alpha\delta}{\lambda};\ \mathrm{Fo} = \frac{a\tau}{\delta^2};\ \mathrm{Bi}_0 = \mathrm{Bi}_c\frac{1 + r_c\alpha_\kappa^*}{1 + r_\kappa^*\alpha_c};\ \mathrm{Bi}_c = \frac{\alpha_c\delta}{\lambda};$$

$$\gamma = \frac{2ar_c r_n^*}{a_\kappa r_\kappa^*(r_c + r_\kappa^*)}\left(\frac{\delta_\kappa}{\delta}\right)^2;$$

where a, a_k represent the thermal conductivity of the plate and the rod; $r_p^* = \frac{2\delta}{\lambda}$ -- thermal resistance of the plate; $r = \frac{r_o}{\delta}$; r_o--polar radius; $R = \frac{R_o}{\delta}$; R_o--hole radius.

If $t|_{\tau=0} = 0$, then--assuming, for example, that the medium temperature is a harmonic function of time, i.e.,

$$t_c = t_0 e^{i\omega\mathrm{Fo}},\ \omega = \mathrm{const}, \tag{14}$$

employing the Laplace transformation, from (12) - (14), we obtain the nonstationary temperature field of the plate in the following form

$$t = t_0 e^{-\mathrm{Mi}} Q_0(r, \mathrm{Fo}, \eta) + t_c\{1 + i\omega[E_0(r, \eta) - E_0(r, \mathrm{Fo}, \eta)]\}, \quad (15)$$

where /141

$$Q_n = \frac{2A}{\pi}\int_0^{\infty} e^{-\eta^2 \mathrm{Fo}} f(r, \eta)\, d\eta, \quad E_n = \frac{2A}{\pi}\int_0^{\infty} e^{-(\eta^2 + i\omega + \mathrm{Bi})\mathrm{Fo}} \frac{f(r, \eta)\, d\eta}{\eta^2 + i\omega + \mathrm{Bi}};$$

$$f(r, \eta) = \frac{J_n(r\eta)[\eta Y_1(R\eta) + BY_0(R\eta)] - Y_n(r\eta)[\eta J_1(R\eta) + BJ_0(R\eta)]}{\eta^{n+1}\{[\eta J_1(R\eta) + BJ_0(R\eta)]^2 + [\eta Y_1(R\eta) + BY_0(R\eta)]^2\}},$$

$A = \mathrm{Bi}_0 - \gamma\,\mathrm{Bi}$; $B = A - \gamma\eta^2$; Mi--Mikheyev condition; $J_n(\eta)$, $Y_n(\eta)$, --Bessel functions of the I and II type of the real argument; n = 0.1.

For an asymptotic thermal regime (Ref. 2), we have the following instead of (15)

$$t^{ac} = \frac{t_c}{\zeta}[SK_0(r\sqrt{\zeta}) + \mathrm{Bi}], \quad (16)$$

where

$$S = \frac{Ai\omega}{\sqrt{\zeta}K_1(R\sqrt{\zeta}) + (\mathrm{Bi}_0 + i\omega\gamma)K_0(R\sqrt{\zeta})}; \quad \zeta = \mathrm{Bi} + i\omega;$$

where K_n is the Macdonald function n = 0.1.

If $t_c = t_0 \cos\omega\,\mathrm{Fo}$, we may find the temperature field by determining the real part of expressions (15), (16), respectively, in the following form

$$t = t_c + t_0 M_0(r, \mathrm{Fo}, \eta); \quad (17)$$

$$t^{ac} = \frac{t_0}{|\zeta|^2}\left\{\frac{1}{h_1^2 + h_2^2}[(\cos\omega\,\mathrm{Fo})(h_1 l_1 + h_2 l_2) - (\sin\omega\,\mathrm{Fo})(h_1 l_2 - h_2 l_1)] + \mathrm{Bi}(\mathrm{Bi}\cos\omega\,\mathrm{Fo} + \omega\sin\omega\,\mathrm{Fo})\right\}, \quad (18)$$

where

$$M_n = \frac{2A}{\pi}\int_0^{\infty}\frac{f(r, \eta)}{(\mathrm{Bi} + \gamma^2)^2 + \omega^2}[\omega^2\cos\omega\,\mathrm{Fo} - \omega(\mathrm{Bi} + \eta^2)\sin\omega\,\mathrm{Fo} - \omega^2 e^{-\mathrm{Mi} - \eta^2\mathrm{Fo}}]\, d\eta, \quad n = 0; \quad l_1 = \mathrm{Bi}U_0 + \omega V_0, \quad l_2 = \mathrm{Bi}V_0 - \omega U_0;$$

$$h_1 = \sqrt{\frac{|\zeta| + \mathrm{Bi}}{2}}U_1^* - \sqrt{\frac{|\zeta| - \mathrm{Bi}}{2}}V_1^* + \omega\gamma U_0^* + \mathrm{Bi}_0 V_0^*;$$

$$h_2 = \sqrt{\frac{|\zeta| + \mathrm{Bi}}{2}}V_1^* + \sqrt{\frac{|\zeta| - \mathrm{Bi}}{2}}U_1^* + \omega\gamma V_0^* - \mathrm{Bi}_0 U_0^*, \quad H^{(1)}(ir\sqrt{\zeta}) = U_n + iV_n, \quad n = 0,1; \quad U_n^* = U_n|_{r=R},$$

$$V_n^* = V_n|_{r=R}.$$

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We should note that we have the following formula in the case of thermally insulated surfaces of the plate ($\alpha = 0$) and the rod ($\alpha_k = 0$) $z = \pm\delta$ instead of (18):

$$t^{ac} = \frac{t_0 Bi_0^*}{d_1^2 + d_2^2}\{d_1 [\cos \omega \, Fo \ker_0 (r\sqrt{\omega}) - \sin \omega \, Fo \, kei_0 (r\sqrt{\omega})] + \\ + d_2 [\cos \omega \, Fo \, kei_0 (r\sqrt{\omega}) + \sin \omega \, Fo \ker_0 (r\sqrt{\omega})]\}, \tag{19}$$

where

$$d_1 = Bi_0^* \ker_0 (R\sqrt{\omega}) - \omega\gamma \, kei_0 (R\sqrt{\omega}) - \\ - \frac{\sqrt{\omega}}{\sqrt{2}}\left[\ker_1 (R\sqrt{\omega}) + kei_1 (R\sqrt{\omega})\right]; \\ d_2 = Bi_0^* kei_0 (R\sqrt{\omega}) + \omega\gamma \ker_0 (R\sqrt{\omega}) - \\ - \frac{\sqrt{\omega}}{\sqrt{2}}\left[kei_1 (R\sqrt{\omega}) - \ker_1 (R\sqrt{\omega})\right]; \\ n = 0,1; \; Bi_0^* = \frac{Bi_c r_c}{r_c + r_K^*}.$$

Setting $\omega = 0$ in (15) we may obtain the solution of the thermal conductivity problem.

$$t = t_0 [1 + e^{-Ml} Q_0 (r, \; Fo, \eta)], \tag{20}$$

which corresponds to the case when the average temperature is the Heavyside H function of time.

In order to determine the stress-deformed state caused by temperature fields (15) -- (20) in the plate, we may employ the condition of simultaneous deformation of the rod and plate.

$$u = \left[\frac{m}{E_K}\sigma_r + \alpha_t^{(k)} t_K\right] R \text{ for } r = R \tag{21}$$

and the expressions which are known from (Ref. 4) for stresses and displacements in a plate

$$\left.\begin{aligned} \sigma_r &= E\left[\frac{c_1}{1-\nu} - \frac{c_2}{r^2(1+\lambda)} - \frac{\alpha t}{r^2}\int_R^r rtdr\right]; \\ \sigma_\theta &= E\left[\frac{c_1}{1-\nu} + \frac{c_2}{r^2(1+\nu)} + \frac{\alpha_t}{r^2}\int_R^r rtdr - \alpha_t t\right]; \\ u &= c_1 r + \frac{c_2}{r} + \frac{\alpha_t}{r}(1+\nu)\int_R^r rtdr, \end{aligned}\right\} \tag{22}$$

where $m = \frac{R_0}{2\delta_k}$; E_k, E -- Young's modulus of the rod and plate; $\alpha_t^{(k)}$, α_t -- their temperature coefficients of linear expansion; ν--Poisson coefficient of the plate. By determining the integration constants c_1, c_2, from condition (21) /143 and from the condition that there is no stress at infinity, in the case of the temperature field (16) we have the following expressions for temperature stresses in the plate;

$$\sigma_r^{ac} = D\left[\alpha_t t_c \frac{Bi}{\zeta} - \alpha_t^{(k)} t_K\right]\left(\frac{R}{r}\right)^2 + \alpha_t t_c E \frac{S}{\zeta}\left[\frac{K_1(r_1)}{r_1} - \right. \tag{23}$$

$$-\left(\frac{R}{r}\right)^2 \frac{K_1(R_1)}{R_1}\Bigg], \quad \sigma_\theta^{ac} = -\sigma_r^{ac} - \alpha_t t_c E \frac{S}{\zeta} K_0(r_1), \tag{23}$$

where

$$D = \frac{1}{\frac{m}{E_\kappa} + \frac{1+\nu}{E}}, \quad r_1 = r\sqrt{\zeta}, \quad R_1 = R\sqrt{\zeta}.$$

In the case of the temperature field (20) we shall have

$$\sigma_r = D\left[\alpha_t t_0^* - \alpha_t^{(k)} t_\kappa\right]\left(\frac{R}{r}\right)^2 + \alpha_t E t_0 e^{-Mi}\left[\left(\frac{R}{r}\right)^2 \frac{Q_1(R, \mathrm{Fo}, \eta)}{R} - \right. \tag{24}$$
$$\left. - \frac{Q_1(r, \mathrm{Fo}, \eta)}{r} + \frac{R^2 - r^2}{2r^2}\right], \quad \sigma_\theta = -\sigma_r + \alpha_t E(t_0^* - t),$$

where

$$t_0^* = t_0(1 - e^{-Mi}).$$

Expressing the stress (24) in Cartesian coordinates and transposing the coordinate origin to the point ($x_1 = R$, $y_1 = 0$), we may strive to the limit in the case $R \to \infty$. As a result, we obtain the stress

$$\sigma_y = -\alpha_t E t_0 \left\{\frac{A}{\sqrt{1 - 4\gamma A}}\left[F_2^-(x) - F_1^-(x) - f_2(x) + f_1(x) + \right.\right. \tag{25}$$
$$\left.\left. + e^{-Mi} \operatorname{erfc}\left(\frac{x}{2\sqrt{\mathrm{Fo}}}\right)\right\},\right.$$

which is produced in a semi-infinite plate with the edge x = 0 reinforced by a thin rod.

Here we have

$$F_i^\pm(x) = \frac{e^{p_i \mathrm{Fo}}}{2\sqrt{\mathrm{Bi} + p_i}}\left[e^{-x\sqrt{\mathrm{Bi}+p_i}} \operatorname{erfc}\left(\frac{x}{2\sqrt{\mathrm{Fo}}} - \sqrt{(\mathrm{Bi} + p_i)\mathrm{Fo}}\right) \pm \right.$$
$$\left. \pm e^{x\sqrt{\mathrm{Bi}+p_i}} \operatorname{erfc}\left(\frac{x}{2\sqrt{\mathrm{Fo}}} + \sqrt{(\mathrm{Bi} + p_i)\mathrm{Fo}}\right)\right],$$

$$f_i(x) = \frac{\mathrm{Bi}_0 + \gamma p_i}{\sqrt{p_i + \mathrm{Bi}}} \cdot F_i^+, \quad \operatorname{erfc}(\eta) = \frac{2}{\sqrt{\pi}}\int_\eta^\infty e^{-\eta^2} d\eta, \quad p_{1,2} =$$
$$= \frac{1 - 2\gamma \mathrm{Bi}_0 \pm \sqrt{1 - 4\gamma A}}{2\gamma^2}.$$

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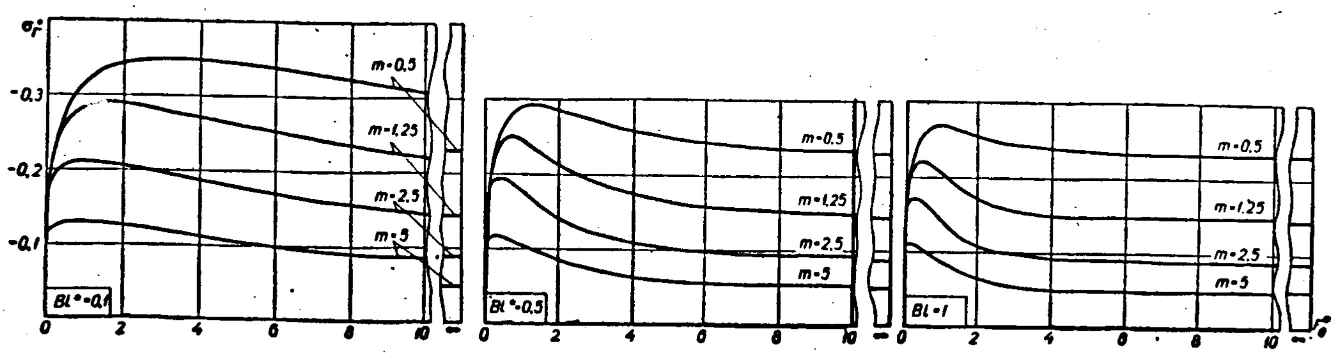
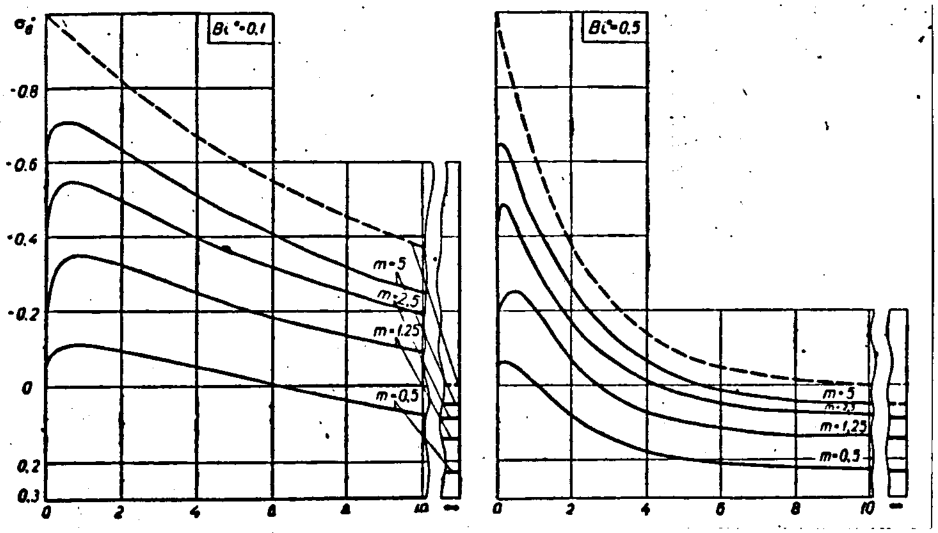


Figure 2



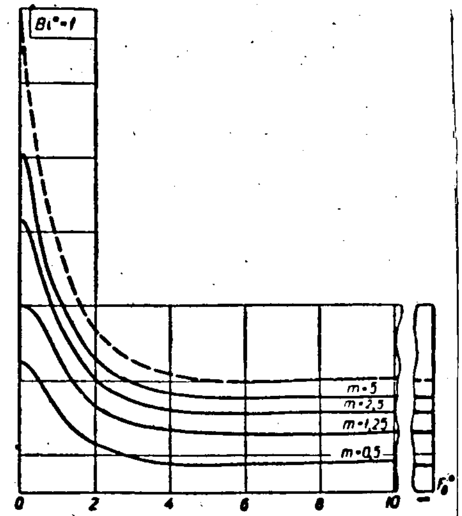


Figure 3

Analysis of Temperature Stresses on the Reinforced Circular Edge of an Infinite Plate for a Nonstationary Thermal Regime /144

On the edge of the plate r = R, and we obtain the following from (24)

$$\sigma_r = D\,[\alpha_t t_0^* - \alpha_t^{(k)} t_K], \quad \sigma_\theta = -\sigma_r + \alpha_t E\,(t_0^* - t_K), \quad t_K = t_0\,[1 - e^{-\mathrm{Mi}} Q_0^{(k)}]. \tag{26}$$

If the edge of the plate r = R is not reinforced ($\alpha_K^* = 0$, $r_K^* = 0$, $\lambda_K^* = 0$, $c_K^* = 0$, $\alpha_t^{(k)} = 0$, $E_K = 0$), we then have the following instead of (26)

$$\tilde{\sigma}_r = 0, \quad \tilde{\sigma}_\theta = \alpha_t E\,(t_0^* - \tilde{t}_K), \quad \tilde{t}_K = t_0\,[1 - e^{-\mathrm{Mi}} \tilde{Q}_0^{(k)}], \tag{27}$$

where

$$Q_0^{(k)} = \frac{4A^*}{\pi^2} \int_0^\infty \frac{e^{-\eta^2 \mathrm{Fo}^*}\, d\eta}{\eta\,\{[\eta J_1(\eta) + B^* J_0(\eta)]^2 + [\eta Y_1(\eta) + B^* Y_0(\eta)]^2\}},$$

$$\tilde{Q}_0^{(k)} = Q_0^{(k)}\big|_{A^* = B^* = \mathrm{Bi}_c}, \quad A^* = AR, \quad B^* = BR, \quad \mathrm{Bi}_c^* = \frac{\alpha_c R_0}{\lambda},$$

$$\mathrm{Fo}^* = \frac{a\tau}{R_0^2}.$$

If $r_c = 0$, we have the following from (26), (27)

$$\sigma_r^* = \frac{1}{m_e + 1 + \nu}\left\{1 - e^{-\mathrm{Mi}} - \alpha_t^*\,[1 - e^{-\mathrm{Mi}} Q_0^{(k)}\big|_{A^* = B^* = m_\lambda}]\right\}, \tag{28}$$

$$\sigma_\theta^* = -\sigma_r^* - e^{-\mathrm{Mi}}\,[1 - Q_0^{(k)}\big|_{A^* = B^* = m_\lambda}]; \tag{29}$$

$$\tilde{\sigma}_r^* = 0, \quad \tilde{\sigma}_\theta^* = -e^{-\mathrm{Mi}}, \tag{29}$$

where

$$\sigma_r^* = \frac{\sigma_r}{\alpha_t E t_0}, \quad \sigma_\theta^* = \frac{\sigma_\theta}{\alpha_t E t_0}, \quad \tilde{\sigma}_r^* = \frac{\tilde{\sigma}_r}{\alpha_t E t_0}, \quad \tilde{\sigma}_\theta = \frac{\sigma_\theta}{\alpha_t E t_0},$$

$$m_E = m\frac{E}{E_\kappa},\ m_\lambda = m\frac{\lambda_\kappa}{\lambda},\ a_t^* = \frac{a_t^{(k)}}{a_t}. \qquad (29)^1$$

Figures 2 and 3 present graphs showing the stress distribution (28) and (29) as a function of Fo*, Bi*, m for a steel (steel carbide) plate, whose circular edge is reinforced by a thin bronze rod (90/10 bronze). When compiling the graphs, we used a graph for the function $Q_0^{(k)}\big|_{A^* = B^* = m_\lambda}$ given in (Ref. 3). The stresses always decrease with an increase in the condition Bi*.

It may be seen from the graph showing the stress distribution σ_r^* (Figure 2) that its maximum values for all m is reached for finite values of Fo*. With an increase in Bi*, the values of Fo* -- corresponding to the maximum values of σ_r^* -- decrease considerably. The maximum values of σ_r^* decrease with an increase in Bi.

The maximum stress values σ_θ^* (Figure 3) for certain large values of m are /146 also reached for finite values of Fo*, decreasing with an increase in Bi* and remaining negative. However, for small m the maximum stress values σ_θ^* are always reached under a stationary thermal regime -- i.e., for an infinitely large value of Fo*.

The solid line in Figure 3 gives the stress in a reinforced plate, and the dashed line gives the stress in an un-reinforced plate;

$$r_c = 0;\ \frac{1}{\lambda_\kappa} = \frac{\lambda_{\text{steel carbide}}}{\lambda_{\text{bronze}}} = 0.5;$$

$$a_t = \frac{a_{t\,\text{bronze}}}{a_{t\,\text{steel carbide}}} = 1{,}5;\ \frac{E}{E_\kappa} = \frac{E_{\text{steel carbide}}}{E_{\text{bronze}}} = 1.75.$$

The author would like to thank Ya. S. Podstrigach for his discussions.

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[1] Translators Note: This is probably incorrect in the original Russian and should be 30, 31.

/M3

3 EFFECT OF CONCENTRATED FORCES IN MULTIPLY CONNECTED REGIONS /147

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The study (Ref. 2) advanced an approximate method for determining the stress state of an elastic isotropic medium weakened by a finite number of circular holes. Distributed stresses were applied to the edges of these holes. This method is extended in this article to the case when the medium is deformed under the influence of concentrated forces applied to the hole edges.

Let us investigate an elastic isotropic medium with two identical circular holes. Concentrated forces are applied to the hole edges, as is shown in Figure 1. We shall assume the hole radius is r = 1, and the distance between the hole centers is 2ℓ.

The problem of the stress state of such a medium may be reduced to determining the functions of a complex variable $\phi(z)$ and $\chi(z)$ from the following boundary conditions on the hole edges (Ref. 8):

$$\varphi(t) + (t - \bar{t})\overline{\varphi'(t)} + \overline{\chi(t)} = f. \tag{1}$$

As is known (Ref. 5), the function f on the arc $\widehat{\sigma_1\sigma_2}$ equals iP, and equals zero on the arc $\widehat{\sigma_2\sigma_1}$.

Just as in (Ref. 8), we shall assume the following representation for the functions $\phi(z)$ and $\chi(z)$:

$$\varphi(z) = \sum_{k=1}^{\infty} \alpha_k \left[\frac{1}{(z-l)^k} + \frac{(-1)^{k+1}}{(z+l)^k}\right], \quad \chi(z) = \sum_{k=1}^{\infty} \beta_k \left[\frac{1}{(z-l)^k} + \frac{(-1)^{k+1}}{(z+l)^k}\right]. \tag{2}$$

Due to the geometric and force symmetry existing in the case under consideration, in order to determine the constants α_k and β_k which are introduced /148 it is sufficient to employ the boundary conditions on the edge of the right hole. They are satisfied automatically on the edge of the left hole. We shall not examine the constants which have no influence on the stress distribution in the medium.

For points on the edge of the right hole, we may represent the functions ϕ and χ as

$$\varphi = \varphi^*(\sigma) + \sum_{k=1}^{\infty} \alpha_k \frac{(-1)^{k+1}}{(\sigma+2l)^k}, \quad \chi = \chi^*(\sigma) + \sum_{k=1}^{\infty} \beta_k \frac{(-1)^{k+1}}{(\sigma+2l)^k}. \tag{3}$$

Here $\phi^*(\sigma)$ and $\chi^*(\sigma)$ represent the boundary values of the functions which are holomorphic in the region outside of the edge of the right hole. We may represent them in the following form:

$$\varphi^*(\zeta) = \sum_{k=1}^{\infty} \alpha_k \zeta^{-k};$$
$$\chi^*(\zeta) = \sum_{k=1}^{\infty} \beta_k \zeta^{-k}, \tag{4}$$

where $\zeta = z - \ell$.

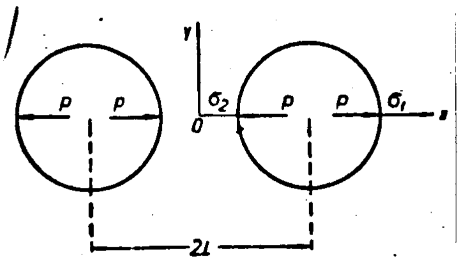

Figure 1

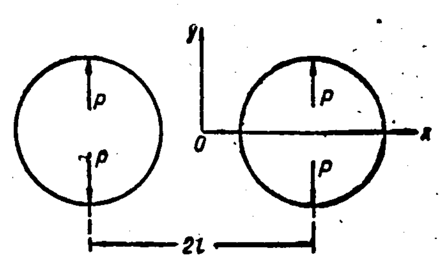

Figure 2

We find the following from the boundary condition (1) on the edge of the right hole, employing the method of N. I. Muskhelishvili:

$$\begin{aligned}\varphi^*(\zeta) &= \frac{P}{2\pi}\ln\frac{\zeta+1}{\zeta-1} + \sum_{k=1}^{\infty}(-1)^k\zeta^{-k}\sum_{m=1}^{\infty}(-1)^m\varepsilon^{k+m}\{\beta_m c^{m-1}_{k+m-1} - \\ &\quad - m\alpha_m(c^m_{k+m-1} - \varepsilon^2 c^m_{k+m+1})\};\\ \chi^*(\zeta) &= -\frac{P}{2\pi}\ln\frac{\zeta+1}{\zeta-1} + (\zeta-\zeta^{-1})\varphi^{*\prime}(\zeta) + \\ &\quad + \sum_{k=1}^{\infty}(-1)^k\delta_k\zeta^{-k}\sum_{m=1}^{\infty}(-1)^m c^{m-1}_{k+m-1}\varepsilon^{k+m}\alpha_m.\end{aligned} \tag{5}$$

Here $\varepsilon = (2\ell)^{-1}$; $\delta_{k=1} = 2$; $\delta_{k>2} = 1$.

An infinite algebraic system having the following form is obtained to determine the coefficients α_k and β_k:

$$\begin{aligned}&\alpha_k + \sum_{m=1}^{\infty}(-1)^{k+m}\{m\alpha_m\varepsilon^{k+m}(c^k_{k+m} - \varepsilon^2 c^{k+1}_{k+m+2}) - \beta^*_m\varepsilon^{k+m}c^k_{k+m-1}\} = f_k;\\ &\beta^*_k + \delta_k\sum_{m=1}^{\infty}(-1)^{k+m+1}\alpha_m c^k_{k+m-1}\varepsilon^{k+m} = t_k,\end{aligned} \tag{6}$$

where

$$\beta^*_k = \beta_k + k\alpha_k - (k-2)\alpha_{k-2},\quad f_k = -t_k = \frac{P}{2\pi k}[(-1)^{k+1}+1]. \tag{7}$$

After determining the constant coefficients α_k and β_k from system /149
(6), the desired functions $\phi(z)$ and $\chi(z)$ assume the following form

$$\begin{aligned}\varphi(z) &= \frac{P}{2\pi}\ln\frac{z-l+1}{z-l-1} + \sum_{k=1}^{\infty}(-1)^k(z-l)^{-k}\sum_{m=1}^{\infty}(-1)^m\varepsilon^{k+m}\{\beta_m c^{m-1}_{k+m-1} - \\ &\quad - m\alpha_m(c^m_{k+m-1} - \varepsilon^2 c^m_{k+m+1})\} + \sum_{k=1}^{\infty}(-1)^{k+1}\alpha_k(z+l)^{-k};\\ \chi(z) &= -\frac{P}{2\pi}\ln\frac{z-l+1}{z-l-1} + \left(z-l-\frac{1}{z-l}\right)\varphi^{*\prime}(z-l) + \\ &\quad + \sum_{k=1}^{\infty}(-1)^{k+1}\beta_k(z+l)^{-k} + \sum_{k=1}^{\infty}(-1)^k(z-l)^{-k}\delta_k\sum_{m=1}^{\infty}(-1)^m c^{m-1}_{k+m-1}\varepsilon^{k+m}\alpha_m.\end{aligned} \tag{8}$$

The stresses produced in the medium may be determined by means of the functions found from the well known formulas (Ref. 8). Let the medium be deformed by concentrated forces, just as is shown in Figure 2. In this case, the functions (5) assume the form

$$\varphi^*(\zeta) = \frac{P}{2\pi i}\ln\frac{\zeta - i}{\zeta + i} + \sum_{k=1}^{\infty}(-1)^k\zeta^{-k}\sum_{m=1}^{\infty}(-1)^m\varepsilon^{k+m}\{\beta_m c_{k+m-1}^{m-1} - m\alpha_m(c_{k+m-1}^{m} - \varepsilon^2 c_{k+m+1}^{m})\};$$
$$\chi^*(\zeta) = \frac{P}{2\pi i}\ln\frac{\zeta - i}{\zeta + i} + \left(\zeta - \frac{1}{\zeta}\right)\varphi^{*\prime}(\zeta) + \sum_{k=1}^{\infty}(-1)^k\alpha_k\delta_k\sum_{m=1}^{\infty}(-1)^m c_{k+m-1}^{m}\varepsilon^{k+m}\zeta^{-m}. \quad (9)$$

The algebraic system for determining the coefficients α_k and β_k is obtained in the form of (6). We thus have

$$f_k = t_k = -\frac{P}{2\pi}\cdot\frac{i^{k-1}}{k}[(-1)^{k+1} + 1]. \quad (10)$$

It must be noted that the left hand sides of equations (6) have the same form as for many other loads of the medium (internal pressure on the hole /150
edges; tension of the medium at infinity; pure shear, etc.).

If the medium is weakened by an infinite series of circular holes, and if concentrated forces are applied to the edges of these holes (Figure 3), then the functions $\phi(z)$ and $\chi(z)$ may be represented in the following form (Ref. 1)

$$\varphi(z) = \sum_{k=1,3,\ldots}^{\infty}\ \sum_{n=-\infty}^{\infty}\frac{\alpha_k}{(z - nl)^k};\quad \chi(z) = \sum_{k=1,3,\ldots}^{\infty}\ \sum_{n=-\infty}^{\infty}\frac{\beta_k}{(z + nl)^k}. \quad (11)$$

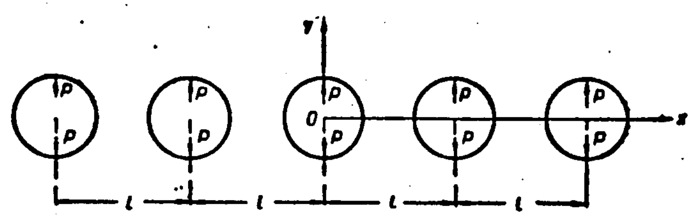

Figure 3

A system is obtained for determining the coefficients α_k and β_k

$$\alpha_k = \sum_{\nu=1,3,\ldots}^{\infty}(-1)^{\nu}\varepsilon^{k+\nu}\{\alpha_{\nu}[\varepsilon^2\lambda_{k+\nu+2}((k+2)c_{k+\nu+1}^{k+2} + \nu c_{k+\nu+1}^{k}) - (k+\nu)\lambda_{k+\nu}c_{k+\nu-1}^{k}] + \beta_{\nu}^*\lambda_{k+\nu}c_{k+\nu-1}^{k}\} = f_k;$$
$$\beta_k^* + \delta_k\sum_{\nu=1,3,\ldots}^{\infty}(-1)^{\nu}\alpha_{\nu}\varepsilon^{k+\nu}\lambda_{k+\nu}c_{k+\nu-1}^{k} = t_k. \quad (12)$$

Here $\lambda_k = 2\sum_{n=0}^{\infty} n^{-k}$, and the values of f_k and t_k are obtained in the form of (10).

If concentrated forces are applied to the hole edges just as is shown in Figure 1, then the left hand sides of system (12) do not change, and the right hand sides assume the form of (7).

In calculating the stress close to any of the hole edges, we must keep the fact in mind that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(z-nl)^k} = \ln\left(1+\frac{1}{z-nl}\right); \qquad \sum_{k=1}^{\infty} \frac{1}{k(z-nl)^k} = -\ln\left(1-\frac{1}{z-ln}\right) \tag{13}$$

$$(n = 1,\ 2,\ \ldots).$$

Formulas (13) make it possible to sum up the poorly converging series and to obtain an effective solution of the problem under consideration.

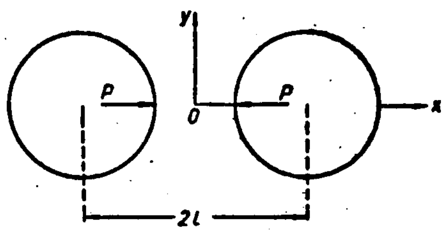

Figure 4

Let us now discuss the case when the medium with two holes, which was examined previously, is deformed by concentrated forces as is shown in Figure 4.[1] /151

In contrast to the previous cases, the principal vector of the external stresses applied to each hole edge will differ from zero. In contrast to expressions (2), the desired functions $\phi(z)$ and $\chi(z)$ will have the following form (Ref. 3):

$$\begin{aligned} \varphi(z) &= \frac{P}{2\pi(1+\varkappa)} \ln\frac{z-l}{z+l} + \sum_{k=1}^{\infty} \alpha_k [(z-l)^{-k} + (-1)^{k+1}(z+l)^{-k}]; \\ \chi(z) &= -\frac{\varkappa P}{2\pi(1+\varkappa)} \ln\frac{z-l}{z+l} + \sum_{k=1}^{\infty} \beta_k [(z-l)^{k} + (-1)^{k+1}(z+l)^{-k}]. \end{aligned} \tag{14}$$

Let us introduce the functions $\phi^*(\zeta)$ and $\chi^*(\zeta)$ in the form of (4). We may determine them by employing the method of N. I. Muskhelishvili. We obtain

$$\begin{aligned} \varphi^*(\zeta) &= \frac{P}{2\pi}\left\{\ln(-1-\zeta^{-1}) - \frac{1}{1+\varkappa}\sum_{k=1}^{\infty}(-1)^{k+1}\zeta^{-k}\varepsilon^k\left(\frac{\varkappa}{k} - \varepsilon^2 + 1\right)\right\} + \\ &+ \sum_{k=1}^{\infty}(-1)^k\zeta^{-k}\sum_{m=1}^{\infty}(-1)^m\varepsilon^{k+m}\{\beta_m C_{k+m-1}^{m-1} - m\alpha_m(C_{k+m-1}^{m} - \varepsilon^2 C_{k+m+1}^{m})\}; \\ \chi^*(\zeta) &= (\zeta - \zeta^{-1})\varphi^{*\prime}(\zeta) + \frac{P}{2\pi(1+\varkappa)}\left\{\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}\varepsilon^k\zeta^{-k} + \varepsilon\zeta^{-1} - \right. \\ &\left. - \zeta^{-2} - (1+\varkappa)\ln(-1-\zeta^{-1})\right\} + \sum_{k=1}^{\infty}\delta_k\zeta^{-k}\sum_{m=1}^{\infty}(-1)^{k+m}C_{k+m-1}^{m-1}\varepsilon^{k+m}. \end{aligned} \tag{15}$$

In order to determine the constant coefficients α_k and β_k just as /152 in the preceding cases, an infinite algebraic system such as (6) is obtained:

$$f_k = \frac{P}{2\pi}(-1)^{k+1}\left\{\frac{1}{k} - \frac{\varepsilon^k}{1+\varkappa}\left(\frac{\varkappa}{k} + 1 - \varepsilon^2\right)\right\}; \quad t_1 = \frac{P}{2\pi}\left(\frac{2\varepsilon}{1+\varkappa} - 1\right); \tag{16}$$

[1] The influence of a concentrated force applied to the edge of one of two circular holes weakening the medium was studied by M. Z. Narodetskiy (Ref. 6) by introducing special functions.

$$t_2 = \frac{P}{2\pi}\cdot\frac{\varkappa - 1 - \varepsilon^2}{2(1+\varkappa)}; \quad t_k = \frac{P}{2\pi}\cdot\frac{(-1)^k}{k}\left(1-\frac{\varepsilon^k}{1+\varkappa}\right) \quad (k > 3), \tag{16}$$

When calculating the stress, the coefficients α_k and β_k may be advantageously represented as follows, within an accuracy of $P/2\pi$:

$$\alpha_k = \frac{(-1)^{k+1}}{k} + \alpha_k'; \quad \beta_k^* = \frac{(-1)^k}{k} + \beta_k^{*\prime}. \tag{17}$$

Then the functions by means of which the stresses are expressed assume the following form

$$\begin{aligned} \varphi(z) &= \frac{P}{2\pi}\left\{\frac{1}{1+\varkappa}\ln\frac{z-l}{z+l} + \ln\left(1+\frac{1}{z-l}\right) - \ln\left(1-\frac{1}{z+l}\right) + \right. \\ &\left. + \sum_{k=1}^{\infty}\alpha_k'\left[\frac{1}{(z-l)^k} + \frac{(-1)^{k+1}}{(z+l)^k}\right]\right\}; \\ \chi(z) &= \frac{P}{2\pi}\left\{-\frac{\varkappa}{1+\varkappa}\ln\frac{z-l}{z+l} + \ln\left(1-\frac{1}{z+l}\right) - \ln\left(1+\frac{1}{z-l}\right) - \right. \\ &\left. -\left(\frac{1}{z-l}+\frac{1}{z+l}\right) + \left[\frac{1}{(z-l)^2} - \frac{1}{(z+l)^2}\right] + \sum_{k=1}^{\infty}\beta_k'\left[\frac{1}{(z-l)^k} + \frac{(-1)^{k+1}}{(z+l)^k}\right]\right\}. \end{aligned} \tag{18}$$

TABLE 1

θ°	l: ∞	4	3	2,5
0	−0,67	−0,00	0,10	0,12
15	−0,58	−0,00	0,10	0,12
30	−0,31	0,00	0,08	0,10
45	0,11	0,01	0,07	0,09
60	0,67	0,04	0,04	0,06
75	1,31	0,10	0,01	0,02
90	2,00	0,23	−0,03	−0,04
105	2,69	0,50	−0,04	−0,11

TABLE 1 (cont.)

θ°	l: ∞	4	3	2,5
120	3,33	1,07	0,06	−0,20
135	3,89	2,16	0,57	−0,28
150	4,31	3,91	2,35	0,09
165	4,58	5,93	6,50	4,85
180	2,67	4,89	7,65	13,45

TABLE 2

Notation	l: 4	3	2,5
$\sigma_x \Big\vert \frac{P}{2\pi}$	−5,40	−11,23	−22,84
$\sigma_y \Big\vert \frac{P}{2\pi}$	1,15	2,49	5,35

By way of an example, we have calculated the stresses σ_θ influencing the surfaces which are normal to the edge of the right hole, for different distances between the holes. Table 1 presents the results of these calculations. The case when $\ell = \infty$ pertains to a medium with one right hole.

In addition, Table 2 presents the results derived from calculations for stresses σ_x and σ_y having an influence at the point O.

It may be seen from these tables that, when the holes converge, the stress /153 concentration sharply increases at the points between the holes, and decreases at the points of the hole edges which are far from the place where the concentrated forces are applied.

Let us now discuss the case when a medium with two circular holes is an anisotropic medium. We shall thus assume that the complex parameters characterizing the medium anisotropy are purely imaginary, i.e., $\mu_1 = i\beta$, $\mu_2 = i\delta$.

The problem of the stress state of such a medium may be reduced, as is known, to determining two functions of the complex variables $\Phi_p(z_p)$ from the following boundary conditions (Ref. 4):

$$2\,\mathrm{Re}\,[\Phi_1(z_2)+\Phi_2(z_1)]=f_1;\quad 2\,\mathrm{Re}\,[\mu_1\Phi_1(z_1)+\mu_2\Phi_2(z_2)]=f_2. \tag{19}$$

Let us transform these conditions to the following form:

$$\Phi_1-k_1\overline{\Phi}_1-k_2\overline{\Phi}_2=-\delta rf,\quad \Phi_2+k_3\overline{\Phi}_1+k_1\Phi_2=\beta rf. \tag{20}$$

Here we have

$$k_1=r(\beta+\delta);\ k_2=2\delta r;\ k_3=2\beta r;\ r=(\beta-\delta)^{-1};\ f=f_1+if_2. \tag{21}$$

Just as in (Ref. 3), we may assume the following representations for the functions $\Phi_p(z_p)(p = 1.2)$

$$\Phi_1=\sum_{k=1}^{\infty}\varphi_k\left\{\frac{1}{[\zeta_1(z_1^*)]^k}+\frac{(-1)^{k+1}}{[\zeta_1(z_1^*+2l)]^k}\right\};$$
$$\Phi_2=\sum_{k=1}^{\infty}\psi_k\left\{\frac{1}{[\zeta_2(z_2^*)]^k}+\frac{(-1)^{k+1}}{[\zeta_2(z_2^*+2l)]^k}\right\}, \tag{22}$$

where

$$z_p^*=z_p-l;\quad z_p^*=m_{0p}\zeta_p+\frac{m_{1p}}{\zeta_p};\quad m_{0p}=0{,}5\,(1-i\mu_p);$$
$$m_{1p}=0{,}5\,(1+i\mu_p). \tag{23}$$

Let the medium be deformed in the same way as is shown in Figure 2. In order to determine the coefficients ϕ_k and ψ_k in the same way as previously, we then obtain the following infinite algebraic system

$$\varphi_k+(m_1^k-k_1)\,A_k^*-k_2B_k^*=P_1\delta\,[1+(-1)^{k+1}]\frac{i^{k-1}}{k};$$
$$\psi_k+(m_2^k-k_1)\,B_k^*+k_3A_k^*=-P_1\beta\,[1+(-1)^{k+1}]\frac{i^{k-1}}{k}. \tag{24}$$

Here we have /154

$$P_1=\frac{Pr}{2\pi};\quad m_p=\frac{m_{1p}}{m_{0p}};\quad A_k^*=\sum_{i=1}^{\infty}(-1)^{i+1}\,\varphi_iA_{ik}\cdot B_k^*=$$
$$=\sum_{i=1}^{\infty}(-1)^{1+i}\psi_iB_{ik}. \tag{25}$$

Thus the coefficients A_{ik} and B_{ik} are obtained when the functions $[\zeta_p(z_p^* + 2\ell)]^{-i}$ are expanded in series with respect to the Fabry polynomials $P(z_p^*)$ within the ellipses obtained from the right ellipse by means of the well known affine transformations

$$[\zeta_1(z_1^*+2l)]^{-i}=\sum_{k=0}^{\infty}A_{ik}P_k(z_1^*);\quad [\zeta_2(z_2^*+2l)]^{-i}=\sum_{k=0}^{\infty}B_{ik}P_k(z_2^*). \tag{26}$$

After the coefficients ϕ_k and ψ_k are determined from system (24), the function $\Phi_p(z_p)$ is known on the basis of (22). The stresses arising in an anisotropic medium may be determined by means of this function as follows

(Ref. 6):

$$\sigma_x = 2\,\mathrm{Re}\,[\mu_1^2\Phi_1' + \mu_2^2\Phi_2']; \quad \sigma_y = 2\,\mathrm{Re}\,[\Phi_1' + \Phi_2']; \quad \tau_{xy} = -2\,\mathrm{Re}\,[\mu_1\Phi_1' + \mu_2\Phi_2']. \tag{27}$$

If concentrated forces are applied in another manner to the hole edges, but the principal stress vector on each hole equals zero, then only the right hand sides of equations (24) need to be changed when solving the problem.

If the principal vector of the external stresses applied to the hole edges does not equal zero on each edge, then the representation of the function (22) must be changed by adding logarithmic terms to it, just as was done in the monograph (Ref. 7).

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DESIGN OF SHELLS OF REVOLUTION WITH A SMALL HOLE AT THE APEX UNDER SYMMETRICAL AND ANTISYMMETRICAL LOAD /155

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The problem of symmetrical and antisymmetrical load for shells of revolution with a small hole at the apex may be solved uniformly.

The angles θ and ϕ are selected as the curvilinear coordinates (Figure 1). The customary dependence of the desired and given quantities on the angle ϕ is assumed:

$$\left.\begin{array}{c} (u,\ w,\ u_\rho,\ \vartheta) = (u_k,\ w_k,\ u_{\rho,k},\ \vartheta_k)\cos k\varphi, \\ v = v_k \sin k\varphi, \\ (T_1,\ T_2,\ M_1,\ M_2,\ Q_\rho) = (T_{1,k},\ T_{2,k},\ M_{1,k},\ M_{2,k},\ Q_{\rho,k})\cos k\varphi); \\ (q_1,\ q_n) = (q_{1,k},\ q_{n,k})\cos k\varphi; \\ q_2 = q_{2,k}\sin k\varphi, \end{array}\right\} \tag{1}$$

so that k = 0 yields a symmetrical case, and k = 1 yields an antisymmetrical case.

1. Shells formed by the revolution of second order curves around their axes of symmetry are investigated. The main radii of curvature for them have the following form

$$R_1 = \frac{R_0}{(1+\gamma\sin^2\theta)^{3/2}}, \quad R_2 = \frac{R_0}{(1+\gamma\sin^2\theta)^{1/2}}; \tag{2}$$

A sphere corresponds to the value of $\gamma = 0$; a paraboloid corresponds to the value $\gamma = -1$; ellipsoids correspond to the value of $\gamma > -1$; and hyperboloids correspond to $\gamma < -1$.

We shall employ the equation obtained by V. V. Novozhilov (Ref. 2) as the initial equation. Under the assumption of relationships (1) this equation is the customary equation for the main complex function /156

$$\begin{gathered} \tilde{T}_k = \tilde{T}_{1,k} + \tilde{T}_{2,k}; \\ \frac{d^2\tilde{T}_k}{d\theta^2} + \left[\left(2\frac{R_1}{R_2} - 1\right)\operatorname{ctg}\theta - \frac{1}{R_1}\cdot\frac{dR_1}{d\theta}\right]\frac{d\tilde{T}_k}{d\theta} + \\ + \frac{R_1}{R_2}\left(1 - 2\frac{R_1}{R_2}\right)\frac{k}{\sin^2\theta}\tilde{T}_k + i\frac{R_1^2}{cR_2}\tilde{T}_k = 0. \end{gathered} \tag{3}$$

We shall not investigate a nonhomogeneous equation here, since no difficulties are usually entailed in finding a particular solution of the problem. For the class of shells under consideration, the main equation assumes the form

$$\begin{gathered} \frac{d^2\tilde{T}_k}{d\theta^2} + \frac{1+2\gamma\sin^2\theta}{1+\gamma\sin^2\theta}\cdot\frac{\cos\theta}{\sin\theta}\cdot\frac{d\tilde{T}_k}{d\theta} - \\ - \frac{k^2}{\sin^2\theta}\cdot\frac{1-\gamma\sin^2\theta}{(1+\gamma\sin^2\theta)^2}\tilde{T}_k + \\ + i\frac{R_0}{c(1+\gamma\sin^2\theta)^{6/2}} = 0. \end{gathered} \tag{4}$$

We may solve it by the method of "the reference" equation (Ref. 1), assuming the following equation as the reference Bessel equation

$$\frac{d^2y}{d\psi^2} + \frac{1}{\psi}\cdot\frac{dy}{d\psi} - \left(i + \frac{n^2}{\psi^2}\right) y = 0, \tag{5}$$

The fundamental solution of this equation may be written in modified Bessel functions of the first and second type:

$$\left.\begin{aligned} y &= \tilde{C}_1 I_n(\psi\sqrt{i}) + \tilde{C}_2 K_n(\psi\sqrt{i}); \\ i^n I_n(\psi\sqrt{i}) &= \mathrm{ber}_n\,\psi + i\,\mathrm{bei}_n\,\psi; \\ i^{-n} K_n(\psi\sqrt{i}) &= \mathrm{ker}_n\,\psi + i\,\mathrm{kei}_n\,\psi, \end{aligned}\right\} \tag{6}$$

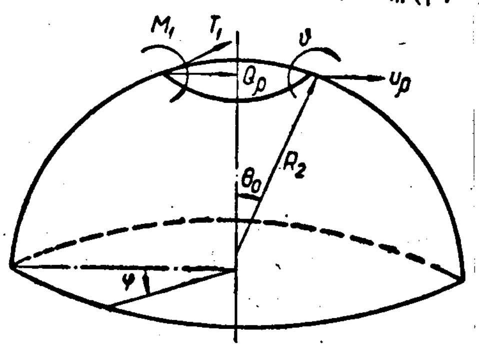

Figure 1

where $\mathrm{ber}_n\,\psi$, $\mathrm{bei}_n\,\psi$, $\mathrm{ker}_n\,\psi$, $\mathrm{kei}_n\,\psi$ are the known tabulated functions of Thomson of the first and second type of the order n.

We shall try to find the solution of equation (4) in the following form

$$\bar{\tilde{T}}_k = \eta(\theta)\, y[\psi(\theta)], \tag{7}$$

where the bar over the complex quantity designates the conjugation sign; y -- solution of the reference equation; $\eta(\theta)$, $\psi(\theta)$ -- the unknown functions to be determined.

Substituting (7) in (4), with allowance for (5), we obtain the following for the latter /157

$$\eta(\theta) = \sqrt{\frac{\psi}{\psi'(1+\gamma\sin^2\theta)^{1/2}\sin\theta}}; \tag{8}$$

$$\psi(\theta) = \lambda\int_0^\theta \frac{d\theta}{(1+\gamma\sin^2\theta)^{5/4}} \tag{9}$$

$$\left(\lambda = \sqrt{\frac{R_2}{c}};\quad c = \frac{h}{\sqrt{12(1-\nu^2)}},\quad h\text{ —shell thickness}\right)$$

and in addition condition n = k.

Since we are interested in the solution close to the small hole at the shell apex, and assuming that the angle θ is sufficiently small, we shall expand the integrand (9) in Taylor series. Performing the integration, we obtain

$$\psi(\theta) = \lambda\theta\,[1 + 0(\theta^2)] \approx \lambda\theta. \tag{10}$$

Neglecting the portion of the solution striving to infinity with an increase in the angle, we may write the desired function in the final form:

$$\tilde{T}_k = \eta(\theta)\,(A - iB)\,(-i)^n\,(\mathrm{ker}_n\,\lambda\theta - i\,\mathrm{kei}_n\,\lambda\theta). \tag{11}$$

When performing differentiation from this point on, we shall disregard the variability of the slowly changing amplitude function $\eta(\theta)$ as compared with the variability of the Thomson function. Figure 2 presents the graph for the function $\eta(\theta)$ for different values of $\gamma \geq 0$.

2. Let us employ $T^*_{1,k}$, $T^*_{2,k}$ to designate the particular solution of the problem. With the value obtained for the fundamental function $\tilde{T}_k$, the complex stresses may be determined according to the following formulas

$$\tilde{T}_{1,k} = T^*_{1,k} + ic\left\{\frac{\operatorname{ctg}\theta}{R_1}\cdot\frac{d\tilde{T}_k}{d\theta} - \frac{k}{R_2\sin^2\theta}\tilde{T}_k\right\};$$
$$\tilde{T}_{2,k} = T^*_{2,k} + \tilde{T}_k - ic\left\{\frac{\operatorname{ctg}\theta}{R_1}\cdot\frac{d\tilde{T}_k}{d\theta} - \frac{k}{R_2\sin^2\theta}\tilde{T}_k\right\}. \tag{12}$$

Thus, the stresses and moments arising in the shell are determined:

$$T_{1,k} = \operatorname{Re}\tilde{T}_{1,k}, \quad T_{2,k} = \operatorname{Re}\tilde{T}_{2,k};$$
$$M_{1,k} = -c\operatorname{Im}\{\tilde{T}_{2,k} + \nu\tilde{T}_{1,k}\}, \quad M_{2,k} = -c\operatorname{Im}\{\tilde{T}_{1,k} + \nu\tilde{T}_{2,k}\}. \tag{13}$$

The following expressions (Ref. 3) are obtained for the boundary intersection force and the bending moment:

$$Q_{\rho,k} = Q^*_{\rho,k} - \frac{c}{\sin\theta}\left\{\frac{1}{R_1}\operatorname{Im}\frac{d\tilde{T}_k}{d\theta} - \frac{k}{R_2}\operatorname{ctg}\theta\operatorname{Im}\tilde{T}_k\right\}; \quad Q^*_{\rho,k} = \cos\theta T^*_{1,k};$$
$$M_{1,k} = -c\left\{\operatorname{Im}\tilde{T}_k - c(1+\nu)\left[\frac{\operatorname{ctg}\theta}{R_1}\operatorname{Re}\frac{d\tilde{T}_k}{d\theta} - \frac{k}{R_2\sin^2\theta}\operatorname{Re}\tilde{T}_k\right]\right\}. \tag{14}$$

Finally, we have the following relationship for radial displacement and the angle of rotation for the shell meridian /158

$$u_{\rho,k} = u^*_{\rho,k} + \frac{R_2\sin\theta}{Eh}\left\{\operatorname{Re}\tilde{T}_k + c(1+\nu)\left[\frac{\operatorname{ctg}\theta}{R_1}\operatorname{Im}\frac{d\tilde{T}_k}{d\theta} - \right.\right.$$
$$\left.\left. - \frac{k}{R_2\sin^2\theta}\operatorname{Im}\tilde{T}_k\right]\right\};$$
$$u^*_{\rho,k} = \frac{R_2\sin\theta}{Eh}(T^*_{2,k} - \nu T^*_{1,k});$$
$$\vartheta_k = -\frac{1}{Eh}\left\{\frac{R_2}{R_1}\operatorname{Re}\frac{d\tilde{T}_k}{d\theta} - k\operatorname{ctg}\theta\operatorname{Re}\tilde{T}_k\right\}. \tag{15}$$

3. Investigating the symmetrical case separately for stresses and moments, we obtain

$$T_1 = T^*_1 + \frac{\eta(\theta)}{\lambda}\operatorname{ctg}\theta(1+\gamma\sin^2\theta)^{3/2}[A\operatorname{kei}'\lambda\theta + B\operatorname{ker}'\lambda\theta];$$
$$T_2 = T^*_2 + \eta(\theta)\left\{A\left[\operatorname{ker}\lambda\theta - \frac{\operatorname{ctg}\theta}{\lambda}(1+\gamma\sin^2\theta)^{3/2}\operatorname{kei}'\lambda\theta - \right.\right.$$
$$\left.\left. - B\left[\operatorname{kei}\lambda\theta - \frac{\operatorname{ctg}\theta}{\lambda}(1+\gamma\sin^2\theta)^{3/2}\operatorname{ker}'\lambda\theta\right]\right]\right\}; \tag{16a}$$
$$M_1 = c\eta(\theta)\left\{A\left[\operatorname{kei}\lambda\theta + (1-\nu)\frac{\operatorname{ctg}\theta}{\lambda}(1+\gamma\sin^2\theta)^{3/2}\operatorname{ker}'\lambda\theta\right] + \right.$$
$$\left. + B\left[\operatorname{ker}\lambda\theta - (1-\nu)\frac{\operatorname{ctg}\theta}{\lambda}(1+\gamma\sin^2\theta)^{3/2}\operatorname{kei}'\lambda\theta\right]\right\}; \tag{16b}$$
$$M_2 = c\eta(\theta)\left\{A\left[\nu\operatorname{kei}\lambda\theta - (1-\nu)\frac{\operatorname{ctg}\theta}{\lambda}(1+\gamma\sin^2\theta)^{3/2}\operatorname{ker}'\lambda\theta\right] + \right.$$
$$\left. + B\left[\nu\operatorname{ker}\lambda\theta + (1-\nu)\frac{\operatorname{ctg}\theta}{\lambda}(1+\gamma\sin^2\theta)^{3/2}\operatorname{kei}'\lambda\theta\right]\right\}. \tag{16d}$$

In addition, we obtain the edge compliance coefficients, using them to designate the radial displacement and edge angle of rotation corresponding to the influence of a single force and moment on the edge -- i.e., assuming the following on the edge /159

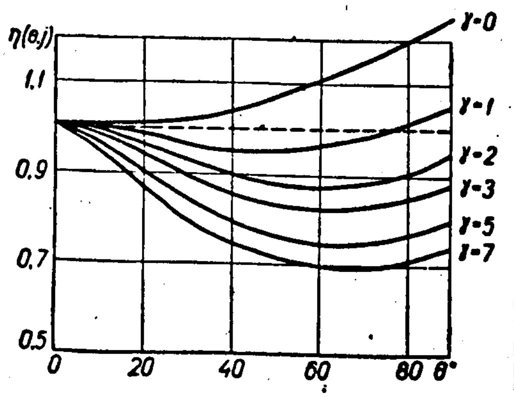

Figure 2

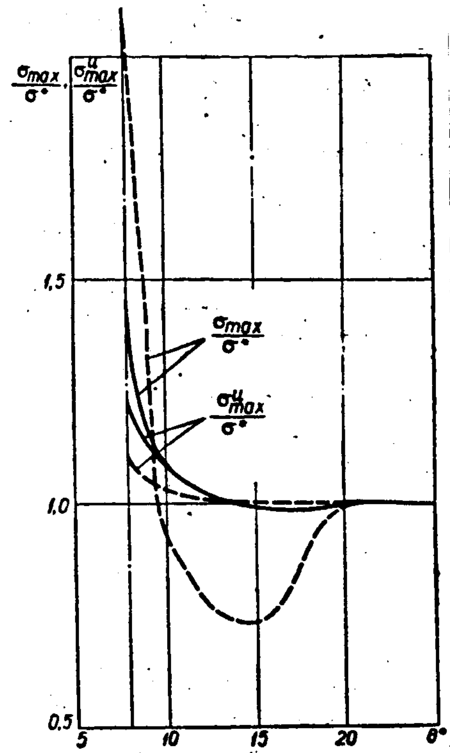

Figure 3

$$\left.\begin{aligned} u_0 &= \overset{*}{u_0} + a_{11}(Q_0 - \overset{*}{Q_0}) + a_{12}M_0; \\ \vartheta_0 &= a_{21}(Q_0 - \overset{*}{Q_0}) + a_{22}M_0, \end{aligned}\right| \qquad (17)$$

where the compliance coefficients have the form

$$\left.\begin{aligned} a_{11} &= \sqrt[4]{12(1-\nu^2)}\,\lambda^3 \frac{\sin^2\theta_0}{(1+\gamma\sin^2\theta_0)^3}\cdot\frac{\Phi_0^2+\Psi_0^2}{X_0}\cdot\frac{1}{E}; \\ a_{12} = a_{21} &= -\sqrt{12(1-\nu^2)}\,\lambda^2 \frac{\sin\theta_0}{(1+\gamma\sin^2\theta_0)^{1/2}}\cdot\frac{\varphi_0\varphi_0'+\psi_0\psi_0'}{X_0}\cdot\frac{1}{Eh}; \\ a_{22} &= [12(1-\nu^2)]^{3/4}\,\lambda\,(1+\gamma\sin^2\theta_0)\frac{\varphi_0'^2+\psi_0'^2}{X_0}\cdot\frac{1}{Eh^2}. \end{aligned}\right\} \qquad (18)$$

Here θ_0 corresponds to the hole edge in the shell:

$$\begin{aligned} \varphi_0 &= \operatorname{ker}\lambda\theta_0; \quad \varphi_0' = \operatorname{ker}'\lambda\theta_0; \\ \psi_0 &= \operatorname{kei}\lambda\theta_0; \quad \psi_0' = \operatorname{kei}'\lambda\theta_0, \end{aligned} \qquad (19)$$

$$\left.\begin{aligned} \Phi_0 &= \varphi_0 - (1+\nu)\frac{\operatorname{ctg}\theta_0}{\lambda}(1+\gamma\sin^2\theta_0)^{3/2}\psi_0'; \\ \Psi_0 &= \psi_0 + (1+\nu)\frac{\operatorname{ctg}\theta_0}{\lambda}(1+\gamma\sin^2\theta_0)^{3/2}\varphi_0'; \\ X_0 &= \Phi_0\psi_0' - \Psi_0\varphi_0'. \end{aligned}\right\} \qquad (20)$$

4. The stresses and moments in the antisymmetrical case assume the following values

$$T_{1,1} = \overset{*}{T}_{1,1} + \frac{\eta(\theta)}{\lambda^2}(1+\gamma\sin^2\theta)^{1/2}\Big\{A\Big[\lambda\operatorname{ctg}\theta\,(1+\gamma\sin^2\theta)\operatorname{ker}_1'\lambda\theta - \\ - \frac{1}{\sin^2\theta}\operatorname{ker}_1\lambda\theta\Big] - B\Big[\lambda\operatorname{ctg}\theta\,(1+\gamma\sin^2\theta)\operatorname{kei}_1'\lambda\theta - \frac{1}{\sin^2\theta}\operatorname{kei}_1\lambda\theta\Big]\Big\}; \qquad (21a)$$

$$T_{2,1} = \overset{*}{T}_{2,1} - \eta(\theta)A\Big\{\operatorname{kei}_1\lambda\theta + \frac{(1+\gamma\sin^2\theta)^{1/2}}{\lambda^2}\Big[\lambda\operatorname{ctg}\theta\,(1+ \\ +\gamma\sin^2\theta)\operatorname{ker}_1'\lambda\theta - \frac{1}{\sin^2\theta}\operatorname{ker}_1\lambda\theta\Big]\Big\} - \eta(\theta)B\Big\{\operatorname{ker}_1\lambda\theta - \\ - \frac{(1+\gamma\sin^2\theta)^{1/2}}{\lambda^2}\Big[\lambda\operatorname{ctg}\theta\,(1+\gamma\sin^2\theta)\operatorname{kei}_1'\lambda\theta - \frac{1}{\sin^2\theta}\operatorname{kei}_1\lambda\theta\Big]\Big\}; \qquad (21b)$$

$$M_{1,1} = c\eta(\theta)A\Big\{\operatorname{ker}_1\lambda\theta - (1-\nu)\frac{(1+\gamma\sin^2\theta)^{1/2}}{\lambda^2}\times$$

$$\times\left[\lambda\,\mathrm{ctg}\,\theta\,(1+\gamma\sin^2\theta)\,\mathrm{kei}_1'\,\lambda\theta-\frac{1}{\sin^2\theta}\mathrm{kei}_1\,\lambda\theta\right]\Big\}-c\eta(\theta)\,B\Big\{\mathrm{kei}_1\,\lambda\theta+$$
$$+(1-\nu)\frac{(1+\gamma\sin^2\theta)^{1/2}}{\lambda^2}\left[\lambda\,\mathrm{ctg}\,\theta\,(1+\gamma\sin^2\theta)\,\mathrm{ker}_1'\,\lambda\theta-\frac{1}{\sin^2\theta}\mathrm{ker}_1\,\lambda\theta\right]\Big\}; \qquad (21c)$$

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$$M_{2,1}=-c\eta(\theta)\,A\Big\{\nu\,\mathrm{ker}_1\,\lambda\theta-(1-\nu)\frac{(1+\gamma\sin^2\theta)^{1/2}}{\lambda^2}\times$$
$$\times\left[\lambda\,\mathrm{ctg}\,\theta\,(1+\gamma\sin^2\theta)\,\mathrm{kei}_1'\,\lambda\theta-\frac{1}{\sin^2\theta}\mathrm{kei}_1\,\lambda\theta\right]\Big\}+$$
$$+c\eta(\theta)\,B\Big\{\nu\,\mathrm{kei}_1\,\lambda\theta+(1-\nu)\frac{(1+\gamma\sin^2\theta)^{1/2}}{\lambda^2}\times\left[\lambda\,\mathrm{ctg}\,\theta\,(1+\gamma\sin^2\theta)\,\mathrm{ker}_1'\,\lambda\theta-\frac{1}{\sin^2\theta}\mathrm{ker}_1\,\lambda\theta\right]\Big\}. \qquad (21d)$$

In this case, the compliance coefficients may be written as follows

$$\left.\begin{aligned}
a_{11}^1&=\frac{1}{\sqrt{12(1-\nu^2)}}\lambda^3\frac{\sin^2\theta_0}{1+\gamma\sin^2\theta_0}\cdot\frac{\Phi_1^2+\Psi_1^2}{X_1}\cdot\frac{1}{E};\\
a_{12}^1&=a_{21}^1=-\lambda^2\frac{\sin\theta_0}{(1+\gamma\sin^2\theta_0)^{1/2}}\cdot\frac{X_2}{X_1}\cdot\frac{1}{Eh};\\
a_{22}^1&=\sqrt{12(1-\nu^2)}\,\lambda\frac{X_3}{X_1}\cdot\frac{1}{Eh^2}.
\end{aligned}\right\} \qquad (22)$$

We employ the following notation here

$$\begin{aligned}
\varphi_1&=\mathrm{ker}_1\,\lambda\theta_0; & \varphi_1'&=\mathrm{ker}_1'\,\lambda\theta_0;\\
\psi_1&=\mathrm{kei}_1\,\lambda\theta_0; & \psi_1'&=\mathrm{kei}_1'\,\lambda\theta_0;
\end{aligned} \qquad (23)$$

$$\Phi_1=\varphi_1-(1+\nu)\frac{(1+\gamma\sin^2\theta_0)^{1/2}}{\lambda^2}\left[\lambda\,\mathrm{ctg}\,\theta_0\,(1+\gamma\sin^2\theta_0)\,\psi_1'-\frac{1}{\sin^2\theta_0}\psi_1\right]; \qquad (24a)$$

$$\Psi_1=\psi_1+(1+\nu)\frac{(1+\gamma\sin^2\theta_0)^{1/2}}{\lambda^2}\left[\lambda\,\mathrm{ctg}\,\theta_0\,(1+\gamma\sin^2\theta_0)\,\varphi_1'-\frac{1}{\sin^2\theta_0}\varphi_1\right]; \qquad (24b)$$

$$\begin{aligned}
X_1&=\Phi_1\left[(1+\gamma\sin^2\theta_0)\,\psi_1'-\frac{\mathrm{ctg}\,\theta_0}{\lambda}\psi_1\right]-\\
&\quad-\Psi_1\left[(1+\gamma\sin^2\theta_0)\,\varphi_1'-\frac{\mathrm{ctg}\,\theta_0}{\lambda}\varphi_1\right];
\end{aligned} \qquad (24c)$$

$$\begin{aligned}
X_2&=\Phi_1\left[(1+\gamma\sin^2\theta_0)\,\varphi_1'-\frac{\mathrm{ctg}\,\theta_0}{\lambda}\varphi_1\right]+\\
&\quad+\Psi_1\left[(1+\gamma\sin^2\theta_0)\,\psi_1'-\frac{\mathrm{ctg}\,\theta_0}{\lambda}\psi_1\right];
\end{aligned} \qquad (24d)$$

$$\begin{aligned}
X_3&=\left[(1+\gamma\sin^2\theta_0)\,\varphi_1'-\frac{\mathrm{ctg}\,\theta_0}{\lambda}\varphi_1\right]^2+\\
&\quad+\left[(1+\gamma\sin^2\theta_0)\,\psi_1'-\frac{\mathrm{ctg}\,\theta_0}{\lambda}\psi_1\right].
\end{aligned} \qquad (24e)$$

5. By setting $\gamma = 0$, $\lambda = \sqrt{\frac{R}{c}}$, $\eta^{(\theta)}=1$, we obtain the case of the sphere with the radius R from the above relationships. We must assume $\lambda = \sqrt{\frac{a^2}{bc}}$ and /161 $\gamma = \frac{a^2}{b^2} - 1$ for an ellipsoidal shell, where a, b are the ellipse semiaxes.

By way of an example, we investigated a shell with a small hole at the apex $\left(\frac{a}{b} = 2,\ \frac{a}{h} = 200,\ \theta = 8°\right)$ under the influence of internal pressure. The curves in Figure 3 give the ratios of the maximum stress $\sigma_{max} = |\sigma^{cat}| + |\sigma_{bend}|$

and the greatest catenary stress to the momentless stress in the shell for a rigidly sealed hole (dashed line) and for a hole reinforced by an elastic ring with a lid (solid line).

6. The homogeneous equation for a symmetrically loaded spherical shell

$$\frac{d^2\tilde{T}}{d\theta^2} + \operatorname{ctg}\theta\frac{d\tilde{T}}{d\theta} + i\frac{R}{c}\tilde{T} = 0 \tag{25}$$

may be reduced to the following by the substitution of $z = \tilde{T}\sqrt{\sin\theta}$:

$$\frac{d^2z}{d\theta^2} + \left[\frac{1}{4}\left(\frac{1}{\sin^2\theta} - \frac{1}{\theta^2}\right) + \frac{1}{4} + \frac{1}{4\theta^2} + i\frac{R}{c}\right]z = 0. \tag{26}$$

Disregarding small terms, we arrive at the Bessel equation and obtain the following solution

$$\tilde{T} = \sqrt{\frac{\theta}{\sin\theta}}(A_1 + iB_1)(\operatorname{ker}\lambda\theta - i\operatorname{kei}\lambda\theta), \tag{27}$$

which is an "accurate" solution in the sense that its error is on the order of $\frac{h}{R}$ as compared with unity.

We obtain the following more accurate formulas for the compliance coefficients

$$\left.\begin{aligned} A_{11} &= \sqrt{12(1-\nu^2)}\left(\frac{R}{h}\right)^2\sin^2\theta_0\frac{W_1^2+W_2^2}{W_0}\cdot\frac{1}{E};\\ A_{12} &= A_{21} = -\sqrt{12(1-\nu^2)}\frac{R}{h}\sqrt{\theta_0\sin\theta_0}\frac{\varphi_0\alpha_0+\psi_0\beta_0}{W_0}\cdot\frac{1}{Eh};\\ A_{22} &= \sqrt{12(1-\nu^2)}\frac{\alpha_0^2+\beta_0^2}{W_0}\cdot\frac{1}{Eh^2}, \end{aligned}\right\} \tag{28}$$

where we employ the following notation /162

$$\left.\begin{aligned} \alpha_0 &= F(\theta_0)\varphi_0 + \sqrt{\frac{R}{c}}\sqrt{\frac{\theta_0}{\sin\theta_0}}\varphi_0';\\ \beta_0 &= F(\theta_0)\psi_0 + \sqrt{\frac{R}{c}}\sqrt{\frac{\theta_0}{\sin\theta_0}}\psi_0';\\ F(\theta) &= \frac{d}{d\theta}\left(\sqrt{\frac{\theta}{\sin\theta}}\right); \end{aligned}\right\} \tag{29}$$

$$\begin{aligned} W_1 &= \sqrt{\frac{\theta_0}{\sin\theta_0}}\psi_0 + (1+\nu)\frac{c}{R}\operatorname{ctg}\theta_0\cdot\alpha_0;\\ W_2 &= \sqrt{\frac{\theta_0}{\sin\theta_0}}\varphi_0 - (1+\nu)\frac{c}{R}\operatorname{ctg}\theta_0\cdot\beta_0;\quad W_0 = \beta_0W_2 - \alpha_0W_1. \end{aligned} \tag{30}$$

7. In order to determine the accuracy of the solution, the compliance coefficients for the sphere, computed according to formulas (18) and according to the customary asymptotic method (Geckeler approximation) were compared with the more accurate coefficients for several values of the wall thinness $\frac{h}{R} = \frac{1}{20}$, $\frac{1}{50}$, $\frac{1}{100}$, $\frac{1}{200}$, $\frac{1}{500}$, and for the angles θ_0 from 5 to 90°.

Figure 4 presents an approximate graph for the compliance coefficients corresponding to the Bessel (solid line) and Geckeler (dashed line) solutions with respect to the more accuract values (assumed to be unity on the graph) for the parameter R/h, equalling 50 and 200. The compliance coefficients

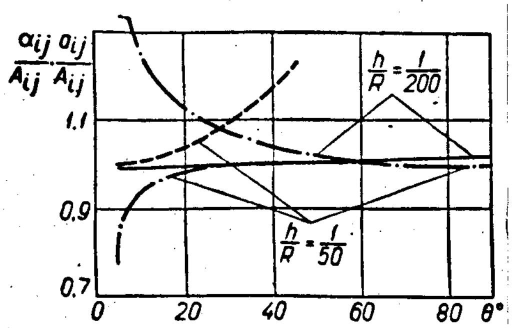

Figure 4

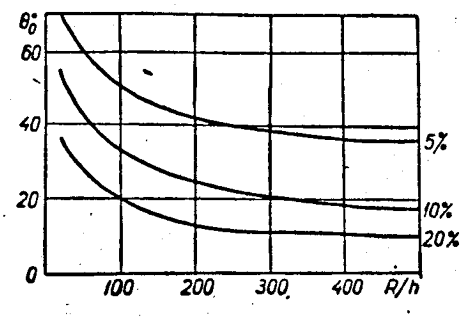

Figure 5

corresponding to the solution (18) were close to the more accurate coefficients. For all values of the parameters, the divergence between them did not exceed 5%. For purposes of comparison, a dashed line was drawn for the value of the compliance coefficients obtained from (18) by substituting $\sin\theta \approx \theta$ in every expression.

Thus, the variational simplification, consisting of substituting the type /163
$\sin\theta \approx \theta$ only in the coefficients of the initial equation, without complicating the calculations, made it possible to widen the range of applicability of the Bessel asymptote up to 90° at least. However, it is not advantageous to employ it at larger angles.

Thus, the graph shown in Figure 5 makes it possible to determine the angle θ_0 beginning with which the Bessel solution may be replaced by the significantly simpler Geckeler solution. The curves shown on the graph correspond to the errors of the Geckeler approximation as compared with the more accurate approximation equalling 5, 10 and 20%.

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/M 3

PROPAGATION OF AN ELASTIC EXPANSION WAVE FROM A CIRCULAR HOLE IN A CYLINDRICALLY ANISOTROPIC, INHOMOGENEOUS PLATE /164

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N67-24522

This article studies the propagation of stress in a thin, infinite plate when a pressure which remains constant is applied to the edge of the circular hole suddenly. It is assumed that the plate material is cylindrically anisotropic (the anisotropy axis coincides with the z-axis of the cylindrical coordinate system r, θ, z, Figure 1) and is continuously inhomogeneous, so that the Young moduli are functions of the radial coordinate

$$E_r = E_1 r^m;\quad E_\theta = E_2 r^m,$$

while

$$\nu_\theta = \text{const};\quad \frac{E_2}{E_1} = \frac{\nu_\theta}{\nu_r} = k;\quad \operatorname{Im}(m) = 0.$$

The static solution of this problem is given in (Ref. 47). The dynamic problem for a homogeneous medium was investigated in (Ref. 5, 7). The article (Ref. 8) studies the propagation of a equivoluminal wave in an isotropic medium with a shear modulus which changes over the radius according to a power law (plane deformation). As will be demonstrated, there is a certain analogy between the present problem and that investigated in (Ref. 8), which was given in (Ref. 6) for a homogeneous isotropic material.

We should point out that the problem for a cylindrical cavity in an infinite medium of the above-mentioned type can be solved in absolutely the same way.

1. If we introduce the following notation /165

$$\bar r = \frac{r}{a};\quad \bar t = \frac{c_0 t}{a};\quad c_0^2 = \frac{E_1 a^m}{(1-\nu_r\nu_\theta)\rho};$$

$$\bar u = \frac{E_1 a^{m-1}}{(1-\nu_r\nu_\theta)\sigma_0} U;\quad \bar\sigma_r = \frac{\sigma_r}{\sigma_0};\quad \bar\sigma_\theta = \frac{\sigma_\theta}{\sigma_0}$$

(where u is the radial displacement; σ_r and σ_θ -- radial and circular normal stresses, a -- hole radius), we then obtain the following equation for radial displacement in the region of Laplace images, with allowance for zero initial conditions;

$$\frac{d^2\bar U}{d\bar r^2} + \frac{m+1}{\bar r}\cdot\frac{d\bar U}{d\bar r} - [p^2\bar r^{2-m} + (k - m\nu_\theta)]\frac{\bar U}{\bar r^2} = 0 \tag{1}$$

and the following boundary conditions

$$\bar r^m\left[\frac{d\bar U}{d\bar r} + \nu_\theta\frac{\bar U}{\bar r}\right] = \begin{cases} -\dfrac{1}{p}, & \bar r = 1, \\ 0, & \bar r \to \infty. \end{cases} \tag{2}$$

Here we have

$$\bar U(\bar r, p) = \int_0^\infty e^{-p\bar t}\bar u(\bar r, \bar t)\, d\bar t \;\dot\rightarrow\; \bar u(\bar r, \bar t).$$

The second condition in (2) is the condition of perturbation damping at infinity. Equation (1) may be reduced to a Bessel equation, and in the case $m \neq 2$ we have the following solution

$$\bar{U} = \bar{r}^{-\frac{m}{2}}\left[A(p)\,K_\nu\left(\frac{p}{1-\frac{m}{2}}\bar{r}^{1-\frac{m}{2}}\right) + B(p)\,I_\nu\left(\frac{p}{1-\frac{m}{2}}\bar{r}^{1-\frac{m}{2}}\right)\right];$$
$$\nu = \frac{1}{2-m}\sqrt{m^2+4(k-m\nu_0)}, \qquad (3)$$

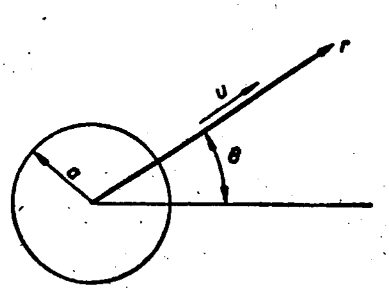

Figure 1

where K_ν, I_ν are the Bessel functions of the imaginary argument.

In the case $m = 2$, we have the Euler equation with the following solution

$$U = A_1(p)\,\bar{r}^{c_1} + B_1(p)\,\bar{r}^{c_2};$$
$$c_{1,2} = -1 \mp \sqrt{p^2+k+1-2\nu_0}. \qquad (4)$$

We should point out that when the medium density ρ changes along the radius according to the power law with an arbitrary real exponent, the solution of the problem does not differ in principle from the solution investigated.

A study of the Bessel function orders (3) shows that they are always real. In the case $m \to \pm\infty$, we have $\nu \to 1$, and for $m \to 2$ we have $\nu \to \infty$.

2. Let us investigate the case $m < 2$. Employing (2), we obtain /166

$$\bar{U} = \bar{r}^{\alpha-1}\frac{\frac{1}{p^2}K_\nu\left(\frac{p}{\alpha}\bar{r}^{\alpha}\right)}{\frac{1-\alpha+\alpha\nu-\nu_0}{p}K_\nu\left(\frac{p}{\alpha}\right)+K_{\nu-1}\left(\frac{p}{\alpha}\right)} = \bar{r}^{\alpha-1}\frac{Z(\bar{r},p)}{N(p)}, \qquad (5)$$
$$\alpha = 1-\frac{m}{2}.$$

Significant difficulties are entailed in directly changing to the class of inverse transforms in (5). Let us rewrite (5) in the following form

$$\bar{U}(\bar{r},\ p)\,e^{\frac{p}{\alpha}}N(p) = e^{\frac{p}{\alpha}}Z(\bar{r},\ p)\,\bar{r}^{\alpha-1}. \qquad (6)$$

The problem now consists of finding the inverse transforms $e^{\frac{p}{\alpha}}K_{\nu-1}\left(\frac{p}{\alpha}\right)$, $e^{\frac{p}{\alpha}}\frac{1}{p}K_\nu\left(\frac{p}{\alpha}\right)$, $e^{\frac{p}{\alpha}}\frac{1}{p^2}K_\nu\left(\frac{p}{\alpha}\bar{r}^{\alpha}\right)$. They were determined in (Ref. 3). The absolute convergence of the Laplace transformations of these inverse transforms may be readily shown. If we employ the following notation

$$e^{\frac{p}{\alpha}}N(p) \mathrel{\dot{\to}} K(\bar{t}),$$

then, according to the convolution theorem, from (6) we obtain the following in the class of inverse transforms

$$\int_0^{\bar{t}}\bar{u}(\bar{r},\ \tau)\,K(\bar{t}-\tau)\,d\tau = z(\bar{r},\ \bar{t})\,\bar{r}^{\alpha-1}. \qquad (7)$$

Differentiating (7) with respect to r, we shall have

$$\int_0^{\bar t} \bar u'(\bar r, \tau) K(\bar t - \tau)\, d\tau = (\alpha - 1) z(\bar r, \bar t)\bar r^{\alpha-2} + z'(\bar r, \bar t)\bar r^{\alpha-1}. \quad (8)$$

Combining (7) and (8), we obtain

$$\int_0^{\bar t} \bar\sigma_r(\bar r, \tau) K(\bar t - \tau)\, d\tau = (\alpha + \nu_\theta - 1) z(\bar r, \bar t)\bar r^{-\alpha} + z'(\bar r, \bar t)\bar r^{1-\alpha}; \quad (9)$$

$$\int_0^{\bar t} \bar\sigma_\theta(\bar r, \tau) K(\bar t - \tau)\, d\tau = (\alpha\nu_\theta + k - \nu_\theta) z(\bar r, \bar t)\bar r^{-\alpha} + z'(\bar r, \bar t)\bar r^{1-\alpha}. \quad (10)$$

Similarly to (7), we may obtain

$$\int_0^{\bar t} \dot{\bar u}(r, \tau) K(\bar t - \tau)\, d\tau = z_1(\bar r, \bar t)\bar r^{\alpha-1}. \quad (11)$$

Here we have /167

$$K(\zeta) = \frac{1-\alpha+\alpha\nu-\nu_\theta}{2\nu}\left[(\alpha\zeta + 1 + \sqrt{(\alpha\zeta+1)^2-1})^\nu - (\alpha\zeta + 1 - \sqrt{(\alpha\zeta+1)^2-1})^\nu\right] + \frac{(\alpha\zeta+1+\sqrt{(\alpha\zeta+1)^2-1})^{\nu-1} + (\alpha\zeta+1-\sqrt{(\alpha\zeta+1)^2-1})^{\nu-1}}{2\sqrt{(\alpha\zeta+1)^2-1}}; \quad (12)$$

$$\begin{aligned} z(\bar r, \bar t) = {} & H\left(\bar t - \frac{\bar r^\alpha - 1}{\alpha}\right)\frac{\bar r^\alpha}{2\alpha\nu(\nu^2-1)}\left\{\left(\frac{\alpha\bar t+1}{\bar r^\alpha} + \sqrt{\frac{(\alpha\bar t+1)^2}{\bar r^{2\alpha}} - 1}\right)^{\nu-1} \times \right. \\ & \times \left[(\nu-1)\frac{\alpha\bar t+1}{\bar r^\alpha}\left(\frac{\alpha\bar t+1}{\bar r^\alpha} + \sqrt{\frac{(\alpha\bar t+1)^2}{\bar r^{2\alpha}} - 1}\right) - 2\nu\right] - \\ & - \left(\frac{\alpha\bar t+1}{\bar r^\alpha} - \sqrt{\frac{(\alpha\bar t+1)^2}{\bar r^{2\alpha}} - 1}\right)^{\nu-1} \cdot \left[(\nu-1)\frac{\alpha\bar t+1}{\bar r^\alpha}\left(\frac{\alpha\bar t+1}{\bar r^\alpha} - \right.\right. \\ & \left.\left.\left. - \sqrt{\frac{(\alpha\bar t+1)^2}{\bar r^{2\alpha}} - 1}\right) - 2\nu\right]\right\}; \end{aligned} \quad (13)$$

$$\begin{aligned} z_1(\bar r, \bar t) = {} & H\left(\bar t - \frac{\bar r^\alpha - 1}{\alpha}\right)\frac{1}{2\nu}\left[\left(\frac{\alpha\bar t+1}{\bar r^\alpha} + \sqrt{\frac{(\alpha\bar t+1)^2}{\bar r^{2\alpha}} - 1}\right)^\nu - \right. \\ & \left. - \left(\frac{\alpha\bar t+1}{\bar r^\alpha} - \sqrt{\frac{(\alpha\bar t+1)^2}{\bar r^{2\alpha}} - 1}\right)^\nu\right]. \end{aligned} \quad (14)$$

In the case $\nu = 1$

$$\begin{aligned} z(\bar r, \bar t) = {} & H\left(\bar t - \frac{\bar r^\alpha - 1}{\alpha}\right)\left[\frac{\alpha\bar t+1}{2\bar r^\alpha}\sqrt{(\alpha\bar t+1)^2 - \bar r^{2\alpha}} - \right. \\ & \left. - \frac{\bar r^\alpha}{2\alpha}\ln\frac{\alpha\bar t+1+\sqrt{(\alpha\bar t+1)^2-\bar r^{2\alpha}}}{\bar r^\alpha}\right]. \end{aligned} \quad (15)$$

Here H ($\bar t$) is the Heavyside function.

Equations (7) - (11) are integral Volterra equations of the first type. Since the right hand parts of all the equations obtained equal zero in the time interval

$$0 < \bar t < \frac{\bar r^\alpha - 1}{\alpha},$$

in equations (7) - (11) we must set the lower limit as $\frac{\bar r^{\alpha} - 1}{\alpha}$.

It is apparent that we have the following in this interval

$$\bar{q} = \bar{u}' = \dot{\bar{u}} = \bar{\sigma}_r = \bar{\sigma}_\theta = 0. \tag{16}$$

At the momentum front we have

$$\bar{t} = \frac{\bar{r}^\alpha - 1}{\alpha}, \tag{17}$$

from which we obtain its propagation rate /168

$$c = \frac{d\bar{r}}{d\bar{t}} = \bar{r}^{1-\alpha} = \bar{r}^{\frac{m}{2}}, \tag{18}$$

or, in the customary notation,

$$c = \sqrt{\frac{E_1 r^m}{(1 - \nu_r \nu_\theta)\rho}}. \tag{19}$$

Substituting $\bar{t} = \frac{r^\alpha - 1}{\alpha} + \varepsilon$ for $\varepsilon \to 0$ in the integral equations, we find that at the momentum front we have

$$\begin{aligned} &\bar{u} = 0;\ \bar{u}' = -\bar{r}^{-\frac{1}{2}-\frac{3}{4}m}; \\ &\bar{\sigma}_r = -\bar{r}^{-\frac{1}{2}+\frac{1}{4}m}; \\ &\bar{\sigma}_\theta = -\nu_\theta \bar{r}^{-\frac{1}{2}+\frac{1}{4}m}; \\ &\dot{\bar{u}} = \bar{r}^{-\frac{1}{2}-\frac{1}{4}m}. \end{aligned} \tag{20}$$

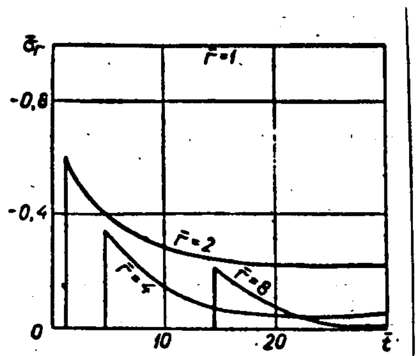

Figure 2

The integral equations obtained may be readily solved numerically, as was done in (Ref. 7), for example.

When the Bessel functions may be expressed in terms of elementary functions, the transition to the space of the inverse transforms is possible by direct inversion of formula (5), employing contour integration and the theorem of residues. Figure 2 presents the results thus obtained for $k = 1$, $\nu_\theta = \frac{1}{4}$, $m \approx 0.7$. It may be readily shown that in the case $t \to \infty$ the stress state in the plate strives asymptotically to the static state.

3. Let us now set $m = 2$. In this case, we may employ the solution of (4) with the following boundary condition

$$\bar{r}^2 \left[\frac{d\bar{U}}{d\bar{r}} + \nu_\theta \frac{\bar{U}}{\bar{r}}\right] = \begin{cases} -\frac{1}{p}, & \bar{r} = 1; \\ 0, & \bar{r} \to \infty. \end{cases}$$

We then have

$$\bar{U}(\bar{r}, p) = \frac{\exp\left(-\ln \bar{r}\sqrt{p^2 + g^2}\right)}{\bar{r}p\left(b + \sqrt{p^2 + g^2}\right)},\ b = 1 - \nu_\theta,\ g^2 = k + 1 - 2\nu_\theta. \tag{21}$$

Since $\dot{\bar{u}}(\bar{r}, \bar{t}) \leftarrow p\bar{U}(\bar{r}, p) = \dot{\bar{U}}$, we have

$$\bar{U}(\bar{r},\ p) = \frac{\exp\{-\ln\bar{r}\sqrt{p^2+g^2}\}}{\bar{r}(b+\sqrt{p^2+g^2})}\cdot\frac{\sqrt{p^2+g^2}}{\sqrt{p^2+g^2}} = \frac{F\left(\sqrt{\left(\frac{p}{g}\right)^2+1}\right)}{\sqrt{\left(\frac{p}{g}\right)^2+1}}, \tag{22}$$

For transformations like (22), there is a formula for transition to the inverse transform which yields /169

$$\dot{\bar{u}}(\bar{r},\ \bar{t}) = f(g\bar{t}) - \int_0^{g\bar{t}} f\left(\sqrt{g^2\bar{t}^2 - z^2}\right) J_1(z)\,dz;\quad f(\bar{t}) \div F(p),$$

where $J_1(z)$ is the Bessel function.

Determining $f(\bar{t})$, we obtain

$$\dot{\bar{u}} = H(\bar{t} - \ln\bar{r})\,\bar{r}^{b-1}\left[e^{-b\bar{t}} - \int_0^{g\sqrt{\bar{t}^2-\ln^2\bar{r}}} \exp\left\{-\frac{b}{g}\sqrt{g^2\bar{t}^2 - z^2}\right\} J_1(z)\,dz\right]; \tag{23}$$

$$\bar{u} = H(\bar{t} - \ln\bar{r})\frac{1}{\bar{r}}\left[\frac{1}{b} - \frac{1}{b}\bar{r}^b e^{-b\bar{t}} - \bar{r}^b \int_{\ln r}^{\bar{t}} d\tau \int_0^{g\sqrt{\tau^2-\ln^2\bar{r}}} \exp\left\{-\frac{b}{g}\sqrt{g^2\tau^2 - z^2}\right\} J_1(z)\,dz\right]; \tag{24}$$

$$\bar{\sigma}_r = H(\bar{t} - \ln\bar{r})\left[-1 + g\ln\bar{r}\int_{\ln\bar{r}}^{\bar{t}} \frac{J_1(g\sqrt{\tau^2-\ln^2\bar{r}})}{\sqrt{\tau^2-\ln^2\bar{r}}}\,d\tau\right]; \tag{25}$$

$$\sigma_\theta = H(\bar{t} - \ln\bar{r})\left\{\frac{b-1+k}{b} + \frac{(b-1)^2-k}{b}\bar{r}^b e^{-b\bar{t}} + [(b-1)^2-k]\,\bar{r}^b\int_{\ln\bar{r}}^{\bar{t}} d\tau \int_0^{g\sqrt{\tau^2-\ln^2\bar{r}}} \exp\left\{-\frac{b}{g}\sqrt{g^2\tau^2 - z^2}\right\} J_1(z)\,dz - g(b-1)\ln\bar{r}\int_{\ln\bar{r}}^{\bar{t}} \frac{J_1(g\sqrt{\tau^2-\ln^2\bar{r}})}{\sqrt{\tau^2-\ln^2\bar{r}}}\,d\tau\right\}. \tag{26}$$

At the momentum front we have

$$\bar{t} = \ln\bar{r},$$

from which we obtain the wave propagation rate $c = e^{\bar{t}} = \bar{r}$, or in the customary notation

$$c = \sqrt{\frac{E_1 r^2}{(1-\nu_r\nu_\theta)\,\rho}}. \tag{27}$$

At the momentum front we have

$$\bar{u} = 0,\quad \dot{\bar{u}} = r^{-1},\quad \sigma_r = -1,\quad \bar{\sigma}_\theta = -\nu_\theta. \tag{28}$$

Figure 3 presents the dependence of $\bar{\sigma}_r$ on $\bar{t}$ for $\nu_\theta = \frac{1}{4}$, $\nu_r = \frac{1}{14}$.

Employing the theorem of the limiting transform value, we have the following from (21) /170

$$\lim_{\bar{t}\to\infty}\bar{u}(\bar{r},\ \bar{t}) = \lim_{p\to 0} p\bar{U}(\bar{r},\ p) = \frac{\bar{r}^{-1+g}}{b+g},$$

which corresponds to the static case. Formula (25) may be rewritten as follows:

$$\bar{\sigma}_r = H(\bar{t} - \ln \bar{r})\left[-1 + g \ln \bar{r} \int\limits_0^{g\sqrt{\bar{t}^2 - \ln^2 \bar{r}}} \frac{J_1(z)}{\sqrt{z^2 + g^2 \ln^2 \bar{r}}}\, dz\right]. \tag{29}$$

Employing the relationship given in (Ref. 1)

$$\int\limits_0^\infty \frac{J_1(x)}{\sqrt{x^2 + a^2}}\, dx = I_{\frac{1}{2}}\left(\frac{1}{2}a\right) K_{\frac{1}{2}}\left(\frac{1}{2}a\right),$$

we obtain the following from (29)

$$\lim_{\bar{t} \to \infty} \bar{\sigma}_r = -\bar{r}^{-g},$$

which also corresponds to the static case.

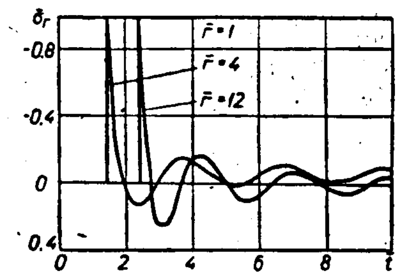

Figure 3

4. In the case m > 2, the argument of the Bessel function in (3) is negative in the case p > 0, and these functions do not acquire real values. In addition, for $\bar{r} \to \infty$ the argument $\frac{p}{\alpha}\bar{r}^\alpha \to 0$, where K_ν has a singularity. Consequently, the condition of perturbation damping by it is not satisfied.

According to (Ref. 8), we shall employ this method to try and obtain a solution in the case under consideration. It is necessary that the second condition in (2) be satisfied only in a finite time interval from the moment that the load is applied. It is natural to only retain the function K_ν in the solution, setting B(p) = 0. The problem may then be reduced in formal terms to the integral equations (7) - (11) with the same kernel and right hand sides. Formulas (16) - (20) also hold.

Let us determine the time interval in which the wave reaches infinity:

$$\bar{t}_\infty = \int\limits_1^\infty \frac{d\bar{r}}{c(\bar{r})} = \frac{1}{1 - \frac{m}{2}} \bar{r}^{1 - \frac{m}{2}} \Big|_1^\infty = \begin{cases} \infty; & m < 2; \\ -\dfrac{1}{1 - \frac{m}{2}}; & m > 2. \end{cases} \tag{30}$$

Thus, in the case m > 2 the wave reaches infinity at a finite moment of time $\bar{t}_\infty$. It follows from (20) that /171

$$\bar{\sigma}_r\left(\bar{t} = \frac{\bar{r}^\alpha - 1}{\alpha} + 0\right) \to \infty; \quad \bar{\sigma}_\theta\left(\bar{t} = \frac{\bar{r}^\alpha - 1}{\alpha} + 0\right) \to \infty \tag{31}$$

in the case $\bar{r} \to \infty$.

Consequently, the stresses at the wave front increase, and become infinitely large during the time $\bar{t}_\infty$. From this time on, the solution loses any physical meaning, and the static stress state is not reached.

Let us examine an example. Let us set k = 1. Then $\nu = -\frac{3}{2}$ in the case $\nu_\theta = \frac{1}{4}$, m $\approx$ 5.7.

We obtain the following from (5)

$$\bar u = \bar r^{-\frac{\alpha}{2}-1}\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{p\left(\bar t-\frac{\bar r^{\alpha}-1}{\alpha}\right)}\frac{p\bar r^{\alpha}+\alpha}{p\left(p^2+\lambda p+\lambda\alpha\right)}\,dp;$$

$$\gamma > 0;\ \lambda = 1-\alpha-\alpha\nu-\nu_0.$$

The integrand satisfies the Jordan lemma in the case $\bar t < -\frac{1}{\alpha}$, and has simple poles at the points $p = 0$ and $p_{1,2} = \phi \pm i\psi$; the latter are roots of the equation $p^2 + \lambda p + \lambda\alpha = 0$. We obtain

$$\bar u = \frac{1}{\lambda}\bar r^{-\frac{\alpha}{2}-1} + \left[\left(\frac{1}{\psi}\bar r^{\frac{\alpha}{2}-1} - \frac{1}{2\psi}\bar r^{-\frac{\alpha}{2}-1}\right)\sin\psi\left(\bar t-\frac{\bar r^{\alpha}-1}{\alpha}\right) - \frac{1}{\lambda}\bar r^{-\frac{\alpha}{2}-1}\cos\psi\left(\bar t-\frac{\bar r^{\alpha}-1}{\alpha}\right)\right]e^{-\frac{\lambda}{2}\left(\bar t-\frac{\bar r^{\alpha}-1}{\alpha}\right)}; \tag{32a}$$

$$\bar\sigma_r = \frac{1-2(\alpha+2)}{4\lambda}\bar r^{-\frac{5}{2}\alpha} + \left[\left(\frac{1-2(\alpha+2)}{4\psi}\bar r^{-\frac{3}{2}\alpha} - \frac{1-2(\alpha+2)}{8\psi}\bar r^{-\frac{5}{2}\alpha} + \frac{\lambda}{2\psi}\bar r^{-\frac{\alpha}{2}}\right)\sin\psi\left(\bar t-\frac{\bar r^{\alpha}-1}{\alpha}\right) + \left(-\bar r^{-\frac{\alpha}{2}} - \frac{1-2(\alpha+2)}{4\lambda}\bar r^{-\frac{5}{2}\alpha}\right)\cos\psi\left(\bar t-\frac{\bar r^{\alpha}-1}{\alpha}\right)\right]e^{-\frac{\lambda}{2}\left(\bar t-\frac{\bar r^{\alpha}-1}{\alpha}\right)}; \tag{32b}$$

$$\bar\sigma_\theta = -\frac{\alpha-6}{8\lambda}\bar r^{-\frac{5}{2}\alpha} + \left[\left(-\frac{\alpha-6}{8\psi}\bar r^{-\frac{3}{2}\alpha} + \frac{\alpha-6}{16\psi}\bar r^{-\frac{5}{2}\alpha} + \frac{\lambda}{8\psi}\bar r^{-\frac{\alpha}{2}}\right)\sin\psi\left(\bar t-\frac{\bar r^{\alpha}-1}{\alpha}\right) + \left(-\frac{1}{4}\bar r^{-\frac{\alpha}{2}} + \tag{32c}$$

$$+\frac{\alpha-6}{8\lambda}\bar r^{-\frac{5}{2}\alpha}\right)\cos\psi\left(\bar t-\frac{\bar r^{\alpha}-1}{\alpha}\right)\right]e^{-\frac{\lambda}{2}\left(\bar t-\frac{\bar r^{\alpha}-1}{\alpha}\right)}; \tag{32d}$$

$$\lambda = -0{,}175,\quad \psi = 0{,}562,\quad \alpha = -1{,}85;\quad \frac{\bar r^{\alpha}-1}{\alpha} < \bar t < -\frac{1}{\alpha}.$$

Figure 4 shows the dependence of σ_r on $\bar t$ for different $\bar r$. /172

The result obtained above shows that, if only a diverging wave ($B(p) = 0$) is taken into account, it is not possible to formulate a solution which is valid in every moment of time.

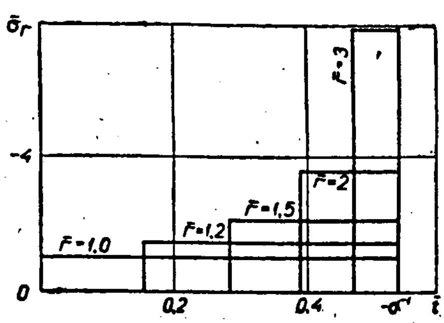

Figure 4

Let us set the following as the boundary condition at infinity

$$\bar U(\bar r,\ p) = 0;\quad \bar r \to \infty, \tag{33}$$

keeping the fact in mind that in the case $\bar r \to \infty$ the material becomes rigid. Then the general solution (3), together with (33a) and the first condition (2), yields

$$A(p) = \frac{2\sin\nu\pi}{\pi}B(p);\quad B(p) = -\frac{\frac{1}{p}}{\frac{2\sin\nu\pi}{\pi}S(p)+T(p)}; \tag{34a}$$

$$\bar{U} = -\bar{r}^{-\frac{m}{2}} \frac{1}{p} \cdot \frac{\frac{2\sin\nu\pi}{\pi} K_\nu\left(\frac{p}{\alpha}\bar{r}^\alpha\right) + I_\nu\left(\frac{p}{\alpha}\bar{r}^\alpha\right)}{\frac{2\sin\nu\pi}{\pi} S(p) + T(p)}; \tag{34b}$$

$$S(p) = (1 - \alpha + \alpha\nu - \nu_0) K_\nu\left(\frac{p}{\alpha}\right) - pK_{\nu-1}\left(\frac{p}{\alpha}\right); \tag{34c}$$

$$T(p) = (1 - \alpha + \alpha\nu - \nu_0) I_\nu\left(\frac{p}{\alpha}\right) + pI_{\nu-1}\left(\frac{p}{\alpha}\right). \tag{34d}$$

Taking the fact into account that

$$K_\nu(z) = \frac{\pi}{2\sin\nu\pi}[I_{-\nu}(z) - I_\nu(z)];$$

$$I_\nu(z) \approx \frac{1}{\Gamma(\nu+1)}\left(\frac{1}{2}z\right)^\nu, \quad z \to 0,$$

we obtain the following from (34)

$$\lim_{\bar{t}\to\infty} u(\bar{r}, \bar{t}) = \lim_{p\to 0} p\bar{U}(\bar{r}, p) = -\frac{\bar{r}^{-\frac{m}{2}-\frac{1}{2}\sqrt{m^2+4(k-m\nu_0)}}}{\frac{m}{2} - \nu_0 + \frac{1}{2}\sqrt{m^2+4(k-m\nu_0)}},$$

which corresponds to the static case.

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/M 3

ONE CONDITION SUFFICIENT FOR THE EXISTENCE OF A SECONDARY LOAD PRODUCING ZERO MOMENT STATE IN A SHELL /174

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Formulation of the problem. Let us investigate a shell having positive curvature; a balanced force field ($\vec{X}$, $\vec{T}$) is applied to the middle surface of this shell. $\vec{X}$ is the external surface load, $\vec{T}$ -- the boundary stress. Let this field be such that the stress state which it produces in the shell differs from a zero moment state. We must determine the secondary load $\vec{X}_0$ which is such that the new force field ($\vec{X} + \vec{X}_0$, $\vec{T}$) produces a zero moment stress state in the shell under consideration.

This article establishes the condition which is sufficient for the existence of a secondary load $\vec{X}$ for a wide class of shells having positive curvature. This load is the generalized potential load given in (Ref. 1, page 575). It is found that, if the edge of the shell which is weakened by one or two holes, contains the arc of any line which is conjugably isometric on the middle surface (Ref. 1, page 122), then the load $\vec{X}_0$ always exists.

Method of solving the problem. In order to solve the problem, we shall employ the method advanced by I. N. Vekua (Ref. 1). As is known, we may write the main system of equations for the zero moment theory of shells as follows (Ref. 1):

$$\nabla_\alpha T^{\alpha\beta} + X^\beta = 0; \quad b_{\alpha\beta} T^{\alpha\beta} + Z = 0; \quad T^{\alpha\beta} = T^{\beta\alpha}\,(\alpha,\, \beta = 1,\, 2), \tag{a}$$

where $T^{\alpha\beta}$ are the contravariant tensor components of stress; Δ_α -- symbol of the covariant differentiation; X^1 and X^2 -- contravariant components of the surface load; Z -- its normal component; $b_{\alpha\beta}$ -- coefficient of the /175 second quadratic form of the middle surface.

If the given shell having positive curvature (K > 0) belongs to a coordinate system which is conjugably isometric, then -- introducing the complex stress function according to the following formula

$$w(\zeta) = u + iv = gK^{1/4}(T^{11} - iT^{12}) + \frac{1}{2} g^{1/2} K^{-1/4} Z \quad (\zeta = x^1 + ix^2), \tag{1}$$

we may reduce the system of equations (a) to the equation given in (Ref. 1):

$$\partial_{\bar{\zeta}} w - \bar{B}\bar{w} = F; \quad \partial_{\bar{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right); \quad \zeta \in G, \tag{2}$$

where

$$B = \frac{1}{4}(\Gamma^1_{22} - \Gamma^1_{11} + 2\Gamma^2_{12}) - \frac{i}{4}(\Gamma^2_{11} - \Gamma^2_{22} + 2\Gamma^1_{12});$$

$$F = \frac{1}{2} g^{1/2} K^{3/4} \partial_\zeta \left(\frac{Z}{K}\right) - K^{3/4}\frac{x^1 - ix^2}{2} g^{1/2};$$

where g is the discriminant of the first quadratic form of the middle surface S, and G is the homeomorphous form of this surface in the ζ-plane. The solutions of equation (2) may be called generalized analytical functions. The complex stress function w may be expressed as follows in terms of the tangential stresses applied to the shell transverse cross-section with the normal $\vec{l}$:

$$w=-\frac{2i}{\zeta'}K^{-1/4}\vec{T}_{(e)}\partial_{\zeta}\vec{n}-\frac{\bar{\zeta}'}{2\zeta'}g^{1/2}K^{-1/4}Z \text{ at } G;\quad \zeta'=\frac{d\zeta}{ds}. \tag{3}$$

However, since the force field ($\vec{X}$, $\vec{T}$) does not produce a zero moment stress state in the shell by definition, at the boundary Γ of the region G the following inequality will hold

$$w\gtrless-\frac{2i}{\tau'}K^{-1/4}\vec{T}\cdot\partial_{\tau}\vec{n}-\frac{\bar{\tau}'}{2\tau'}g^{1/2}K^{-1/4}Z;\quad \tau=\zeta\in\Gamma. \tag{b}$$

We shall now try to determine the supplementary load:

$$\begin{gathered}\vec{X}_0\equiv X_0^{\alpha}\vec{r}_{\alpha}+Z_0\vec{n}=\frac{1}{2}\sqrt{\frac{K}{g}}\cdot\frac{\partial}{\partial x^1}\left(\frac{Z_0}{K}\right)\vec{r}_1+\\+\frac{1}{2}\sqrt{\frac{K}{g}}\cdot\frac{\partial}{\partial x^2}\left(\frac{Z_0}{K}\right)\vec{r}_2+Z_0\vec{n}\end{gathered} \tag{4}$$

where ($\vec{r}_1$ and $\vec{r}_2$ are the fundamental base vectors on the surface S, and $\vec{n}$ are the orthonormals to it), so that the new force field ($\vec{X}+\vec{X}_0,\vec{T}$) produces a zero moment state in the shell. Then at the boundary Γ of region G we obtain the following, instead of inequality (b)

$$w=-\frac{2i}{\tau'}K^{-1/4}\vec{T}\partial_{\tau}\vec{n}-\frac{\bar{\tau}'}{2\tau'}g^{1/2}K^{-1/4}(Z+Z_0);\quad \tau\in\Gamma. \tag{5}$$

It may be readily seen that a determination of the function Z_0, and also of the supplementary load $\vec{X}_0$, leads to the solution of the generalized boundary value problem of Riemann-Hilbert: /176

$$\partial_{\bar{\zeta}}w-B\bar{w}=F;\quad \zeta\in G;\quad \mathrm{Re}\,[i\tau'^2w]=2K^{-1/4}\vec{T}\cdot\frac{d\vec{n}}{ds'};\ \tau\in\Gamma. \tag{A}$$

In order to clarify the solvability of this problem, let us investigate the correlated problem:

$$\partial_{\bar{\zeta}}w^*+B\bar{w}^*=0;\quad \zeta\in G;\quad \mathrm{Re}\,[i\bar{\tau}'w^*]=0,\quad \tau\in\Gamma. \tag{A*}$$

In addition, let us employ n and n* respectively, to designate the indices of the boundary values problems (A) and (A*) (Ref. 1, 2, 3). For a shell weakened by one hole, we shall then have n = − 2, n* = 1. If the shell has two holes, then n = n* = 0. Consequently, the well known conditions of conjugation (Ref. 1, pages 180, 596) must be satisfied in order that there may be a solution of the inhomogeneous problem (A), since the indices of this problem are not positive. These conjugation conditions are satisfied *each time* that one of the shell holes contains an arc of any line which is conjugably isometric on the middle surface.

Let us prove this statement. We shall employ Γ_0 to designate that portion of the boundary for the region G which is a homeomorphic form of the arc given above on the surface S. Let $x^1 = x^1(s)$; x^2 = const be the equation of the curve

Γ_0. Then the boundary condition for the problem (A*) may be assumed to have the form $\mathrm{Re}[i\bar{\tau}'\omega^*] = -\frac{dx^1}{ds}\ \mathrm{Im}\ \omega^* = 0$ (on Γ_0) on this curve. It follows that $\omega^* \equiv 0$; $\zeta \in G$ from this relationship and the uniqueness theory of Kirleman (Ref. 1, page 158). This substantiates the validity of our statement. Thus, with the given assumptions regarding the shell edge, the inhomogeneous problem of Riemann-Hilbert (A) is always solvable.

Employing equation (5), we may now readily find the boundary value of the normal component of the supplementary load $\vec{X}_0$ (4):

$$Z_0 = -(Z + 2\tau'^2 g^{-1/2} K^{1/4} w^+(\tau) + 4i\tau' g^{-1/2} \vec{T}\frac{d\vec{n}}{ds}; \quad \tau \in \Gamma, \tag{6}$$

where ω^+ is the boundary value of the solution ω of the inhomegeneous problem (A). The plus sign over w indicates that the limiting value of the function w is chosen when the point ζ strives to the profile Γ inside the region G. Since in the general solution of the Riemann-Hilbert problem (A) there are no nontrivial, linearily independent homogeneous solutions (since the correlated problem (A*) does not have one solution which differs from zero either for a simply connected or for a doubly connected region), we may extend the function Z_0 (6) continuously within the region G uniquely: /177

$$Z_0 = -(Z + 2\zeta'^2 g^{-1/2} K^{1/4} w^+(\zeta) + 4i\zeta' g^{-1/2} T\frac{d\vec{n}}{ds}; \quad \zeta \in G. \tag{7}$$

Substituting (7) in (4), we may find the explicit expression for the supplementary load $\vec{X}_0$ by means of which a zero moment state is produced in the shell. It follows from the above statements that there is only one correcting load (4) for the shells under consideration. We shall investigate below the spherical shell with a curvilinear hole, and we shall present the analytical expression for the function w^+ in terms of which the load $\vec{X}_0$ may be determined.

Examples. Let us investigate a spherical shell with a circular hole which is loaded by surface and edge stresses. We shall select the parallels and meridians as the lines on the sphere which are isometric in conjugate terms. A circle is then the homeomorphic form of this shell in the plane $\zeta = x^1 + ix^2$. It may then be mapped onto the exterior of the unit circle G+.

We shall employ G⁻ to designate the complementary minor G+ up to the entire ζ-plane. We may then represent the problem (A) for our shell in the following form:

$$\partial_{\bar{\zeta}} w - 0\ (\text{в } G); \quad \mathrm{Re}[i\tau'^2 w] = 2K^{-1/4}\vec{T}\frac{d\vec{n}}{ds} - \mathrm{Re}[i\tau'^2 w_0]; \quad \tau \in \Gamma, \tag{A$_0$}$$

where w_0 is the special solution of the inhomogeneous equation $\partial_{\bar{\zeta}} w_0 = F$ (for the sphere $B \equiv 0$ [Ref. 1]).

Consequently, the determination of the function Z_0 for a spherical shell leads to a solution of the boundary value problem of Riemann-Hilbert for the customary analytical functions

$$\mathrm{Re}\,[(a(\tau) + ib(\tau))\,w] = c(\tau); \quad a + ib = i\tau'^2; \tag{8}$$

$$c = 2K^{-1/4}\vec{T}\cdot\frac{d\vec{n}}{ds} - \mathrm{Re}\,[i\tau'^2 w_0]; \quad \tau \in \Gamma. \tag{8}$$

As is known, we may replace this problem by the equivalent conjugate problem (Ref. 3)

$$w^+(\tau) = D(\tau)\,w^-(\tau) + d(\tau) \quad (\text{on } \Gamma);$$
$$D = -\frac{a-ib}{a+ib}, \quad d = \frac{2c}{a+ib}. \tag{9}$$

Here $w^-(\zeta) = w^+\left(\frac{1}{\bar{\zeta}}\right)$ is the function which is holomorphic in the region G^-. Since the index of the problem (9) is negative, its solution may be represented in the following form:

$$w = \frac{X(\zeta)}{\pi i}\int_\Gamma \frac{c}{(a+ib)X^+(\tau)}\cdot\frac{d\tau}{\tau-\zeta}, \tag{10}$$

where $X(\zeta)$ is the canonical function corresponding to the conjugate problem (9):

$$X(\zeta) = Ce^{\frac{1}{2\pi i}\int_\Gamma \frac{\ln[\tau^2 D(\tau)]}{\tau-\zeta}d\tau}.$$

Thus, the components of the correcting load $\vec{X}_0$ are obtained in explicit form by means of formulas (5), (10), (4). /178

Let us examine the case when the spherical shell is weakened by an arbitrary curvilinear hole without corner points. Then a certain region G which is defined by the profile Γ will be the homeomorphic form of the middle surface of this shell in the ξ-plane. Let $\zeta = \zeta(\omega)$ be the relationship which performs the conformal mapping of the unit circle $|\omega| < 1$ in the ω-plane onto the region G in the ζ-plane. The boundary value problem (A_0) then assumes the following form

$$\partial_{\bar{\omega}} w = 0 \ (\text{in circle } |\omega| < 1); \quad \mathrm{Re}\,[\alpha(\sigma)\,w] = \beta(\sigma); \quad \sigma \in \gamma, \tag{11}$$

where

$$\alpha(\sigma) = i\tau'^2(\sigma); \; \beta(\sigma) = 2K^{-1/4}\vec{T}\frac{d\vec{n}}{ds^*}\cdot\frac{ds^*}{ds} - \mathrm{Re}\,[i\tau'^2 w_0];$$

where σ is a point on the circle γ - $|\sigma| = 1$; ds^* -- an arc element of this circle. Consequently, we may apply this same method of solution to the boundary value problem (A_0) as to the problem (8), corresponding to a spherical shell with a circular hole.

In conclusion, we would like to note that the homeomorphic form G^+ of a spherical shell in the ζ-plane may have the form of a triangle, a trapezoid, a rectangle, or another polygon with rounded apexes. The study (Ref. 4) presents the explicit expressions for the conformal mappings of these regions onto a circle.

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GENERALIZATION OF THE GRIFFITH-SNEDDON CRITERION TO THE CASE OF AN INHOMOGENEOUS ELASTIC BODY /179

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The study by Griffith (Ref. 6), which is devoted to the formation and development of cracks in a brittle body, employs the energy approach. The main assumptions advanced by the Griffith theory state that stress forces, which are similar to forces influencing the surface of a liquid, influence the surface of a solid body, and that the decrease in the potential energy of the body W when a crack is formed, having the length 2a, is balanced by the increase in the surface energy of the crack U.

The necessary condition for the crack increase is

$$\frac{\partial}{\partial a}(W-U)=0. \qquad (1)$$

Griffith obtained a formula for the critical breaking point when an infinite plate with a rectilinear crack having the length 2a is subjected to tension by forces which are perpendicular to the line of the crack

$$p_0=\sqrt{\frac{2ET}{\pi a(1-\nu^2)}}, \qquad (2)$$

where E is the Young's modulus; ν -- Poisson coefficient; T -- surface stress of the material.

Sneddon (Ref. 4) generalized the perturbation theory of Griffith to the three-dimensional case. It was shown in (Ref. 4) that a body with a circular plane crack having the radius a is perturbed when the disruptive stress p, which is normal to the crack plane, exceeds the critical value p_0, and we have

$$p_0=\sqrt{\frac{\pi ET}{2a(1-\nu^2)}}. \qquad (3)$$

Based on the Griffith concept, the present article establishes the criterion /180 for the perturbation of the inhomogeneous elastic body.

Formulation of the problem and derivation of boundary conditions. We shall investigate the problem of the elasticity theory concerning two elastic halfspaces with different elastic properties. In the plane connecting these halfspaces there is a circular crack having the radius a. The tensile stresses p = const. are applied at infinity; these stresses are perpendicular to the crack plane. We shall introduce the rectangular Cartesian coordinates in such a way that the boundary of the elastic halfspaces coincides with the z = 0 plane. We shall place the origin at the crack center.

We may write the solution of the problem in the following form

$$u=\varphi_1+z\frac{\partial\psi}{\partial x};\quad v=\varphi_2+z\frac{\partial\psi}{\partial y};\quad w=\varphi_3+z\frac{\partial\psi}{\partial z}, \qquad (4)$$

where u(x, y, z), v(x, y, z), w(x, y, z) are the projections of the elastic displacements on the axis of the rectangular coordinates; ϕ_1, ϕ_2, ϕ_3, ψ -- the functions of x, y, z which are harmonic in space and which are related by the relationship

$$\frac{\partial\psi}{\partial z}=\frac{1}{4\nu-3}\left(\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial y}+\frac{\partial\varphi_3}{\partial z}\right). \quad (5)$$

We shall employ the index + to designate the functions pertaining to the upper halfspace, and the index - to designate the functions pertaining to the lower halfspace. In order to determine the unknowns of the functions we have the boundary conditions:

$$\left.\begin{aligned}\sigma_z^+(x,y,z)&=\sigma_z^-(x,y,z);\\ \tau_{xz}^+(x,y,z)&=\tau_{xz}^-(x,y,z);\\ \tau_{yz}^+(x,y,z)&=\tau_{yz}^-(x,y,z);\end{aligned}\right\}\ \text{Inside the crack } \rho=\sqrt{x^2+y^2}<a; \quad (6)$$

$$\left.\begin{aligned}u^+(x,y,z)&=u^-(x,y,z); & \sigma_z^+(x,y,z)&=\sigma_z^-(x,y,z);\\ v^+(x,y,z)&=v^-(x,y,z); & \tau_{xz}^+(x,y,z)&=\tau_{xz}^-(x,y,z);\\ w^+(x,y,z)&=w^-(x,y,z); & \tau_{yz}^+(x,y,z)&=\tau_{yz}^-(x,y,z);\end{aligned}\right\}\ \text{Outside the crack } \rho=\sqrt{x^2+y^2}>a. \quad (7)$$

Expressing the stress in terms of deformation, employing (4), and introducing the notation

$$\begin{aligned}\frac{\partial\varphi_4^+(x,y,z)}{\partial z}&=\frac{\partial\varphi_1^+(x,y,z)}{\partial x}+\frac{\partial\varphi_2^-(x,y,z)}{\partial y};\\ \frac{\partial\varphi_4^-(x,y,z)}{\partial z}&=\frac{\partial\varphi_1^-(x,y,z)}{\partial x}+\frac{\partial\varphi_2^-(x,y,z)}{\partial y},\end{aligned} \quad (8)$$

we obtain the boundary conditions for determining the functions ϕ_3 and ϕ_4 in the following form:

$$\left.\begin{aligned}\left[\frac{\partial\varphi_3^-(x,y,z)}{\partial z}-\frac{1}{A_1}\cdot\frac{\partial\varphi_4^-(x,y,z)}{\partial z}\right]_{z=0}&=C;\\ \varphi_3^-(x,y,0)-A_1\varphi_4^-(x,y,0)&=C;\end{aligned}\right\}\ \text{Inside the crack } \rho<a; \quad (9)$$

$$\left.\begin{aligned}\left[\frac{\partial\varphi_4^+(x,y,z)}{\partial z}=\frac{\partial\varphi_4^-(x,y,z)}{\partial z}\right]_{z=0};\\ \varphi_3^+(x,y,0)=\varphi_3^-(x,y,0);\end{aligned}\right\}\ \text{Outside the crack } \rho>a; \quad (10)$$

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$$\left.\begin{aligned}D_2\frac{\partial^2\varphi_3^+}{\partial z^2}-D_1\frac{\partial^2\varphi_3^-}{\partial z^2}-B_2\frac{\partial^2\varphi_4^+}{\partial z^2}-B_1\frac{\partial^2\varphi_4^-}{\partial z^2}&=0;\\ B_2\frac{\partial\varphi_3^+}{\partial z}-B_1\frac{\partial\varphi_3^-}{\partial z}-D_2\frac{\partial\varphi_4^+}{\partial z}-D_1\frac{\partial\varphi_4^-}{\partial z^2}&=0;\end{aligned}\right\}\ \text{On the entire plane } z=0,\ -\infty<\rho<\infty. \quad (11)$$

We employ the following notation

$$A_1=\frac{\lambda_1+2\mu}{\mu_1};\quad C_1=\frac{\lambda_1+3\mu_1}{2\mu_1(\lambda_1+2\mu_1)}p; \quad (12)$$

$$B_1=\frac{\mu_1(\lambda_1+2\mu_1)}{\lambda_1+3\mu_1};\quad B_2=\frac{\mu_2(\lambda_2+2\mu_2)}{\lambda_2+3\mu_2};\quad D_1=\frac{\mu_1^2}{\lambda_1+3\mu_1};\ D_2=\frac{\mu_2^2}{\lambda_2+3\mu_2}; \quad (12)$$

where λ_1, λ_2, μ_1, μ_2 are the Lamé coefficients for the lower and upper half-spaces, respectively; C -- constant to be determined.

Let us set the functions $\phi_3^*(x, y, z)$ and $\phi_4^*(x, y, z)$ in the lower half-space, which equal $\phi_3^+(x, y, z)$ and $\phi_4^+(x, y, z)$, respectively, at the boundary (z = 0), i.e.,

$$\varphi_3^*(x, y, 0) = \varphi_3^+(x, y, 0); \quad \varphi_4^*(x, y, 0) = \varphi_4^+(x, y, 0). \tag{13}$$

We then have

$$\left[\frac{\partial\varphi_3^*(x, y, z)}{\partial z} = -\frac{\partial\varphi_3^+(x, y, z)}{\partial z}\right]_{z=0}; \quad \left[\frac{\partial\varphi_4^*(x, y, z)}{\partial z} = -\frac{\partial\varphi_4^+(x, y, z)}{\partial z}\right]_{z=0};$$
$$\left[\frac{\partial^2\varphi_3^*(x, y, z)}{\partial z^2} = \frac{\partial^2\varphi_3^+(x, y, z)}{\partial z^2}\right]_{z=0}; \quad \left[\frac{\partial^2\varphi_4^*(x, y, z)}{\partial z^2} = \frac{\partial^2\varphi_4^+(x, y, z)}{\partial z^2}\right]_{z=0}. \tag{14}$$

Employing (13) and (14), we obtain the following from relationships (11)

$$\varphi_3^+(x, y, 0) = K\varphi_3^-(x, y, 0) - H\varphi_4^-(x, y, 0);$$
$$\varphi_4^+(x, y, 0) = H\varphi_3^-(x, y, 0) - K\varphi_4^-(x, y, 0), \tag{15}$$

where

$$K = \frac{B_1B_2 + D_1D_2}{D_2^2 - B_2^2}; \quad H = \frac{B_1D_2 + B_2D_2}{D_2^2 - B_2^2}. \tag{16}$$

Taking (13), (14) and (15) into account, we may transform the boundary conditions outside of the crack (10) to the following form

$$\varphi_3^-(x, y, 0) - A_0\varphi_4^-(x, y, 0) = 0;$$
$$\left[\frac{\partial\varphi_3^-(x, y, z)}{\partial z} - \frac{1}{A_0}\cdot\frac{\partial\varphi_4^-(x, y, z)}{\partial z}\right]_{z=0} = 0. \tag{17}$$

Here we have

$$A_0 = \frac{H}{K-1}.$$

Introducing the functions /182

$$F_1(x, y, z) = \varphi_3^-(x, y, z) - A_1\varphi_4^-(x, y, z);$$
$$F_2(x, y, z) = \varphi_3^-(x, y, z) - \frac{1}{A_1}\varphi_4^-(x, y, z), \tag{18}$$

we obtain the following problem of the potential theory for these functions

$$F_1(x, y, 0) = C; \quad \left[\frac{\partial F_2(x, y, z)}{\partial z}\right]_{z=0} = C_1; \quad \rho < a; \tag{19}$$

$$F_1(x, y, 0) - AF_2(x, y, 0) = 0; \quad \left[\frac{\partial F_1(x, y, z)}{\partial z} - B\frac{\partial F_2(x, y, z)}{\partial z}\right]_{z=0} = 0 \quad \rho > a, \tag{20}$$

where we employ the notation

$$A = \frac{A_1 - A_0}{1 + A_1A_0}A_1; \quad B = \frac{1 + A_1A_0}{A_1 - A_0}A_1.$$

Reduction of the axisymmetric problem of the potential theory to the plane problem. The harmonic functions $F_1(x, y, z)$ and $F_2(x, y, z)$, which satisfy the boundary conditions (19), in view of the fact that they are not dependent on the angle ϕ, may be designated by $F_1(\rho, z)$ and $F_2(\rho, z)$, respectively, and we may represent them in the following form

$$F_k(\rho, z) = \int_0^\infty f_k(\alpha)\, I_0(\rho\alpha)\, e^{\alpha z} d\alpha \quad (k = 1, 2). \tag{21}$$

Differentiating (21) with respect to z, we obtain

$$\frac{\partial F_k(\rho, z)}{\partial z} = \int_0^\infty f_k(\alpha)\, \alpha I_0(\rho\alpha)\, e^{\alpha z} d\alpha \quad (k = 1, 2). \tag{22}$$

Let us represent the Bessel function in the form of the boundary integrals

$$\begin{aligned} I_0(\rho\alpha) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{-s}\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}s\right)} \rho^{s-1}\alpha^{s-1} ds; \\ \alpha I_0(\rho\alpha) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{1-s}\Gamma\left(1-\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}s\right)} \rho^{s-2}\alpha^{s-1} ds. \end{aligned} \tag{23}$$

Substituting these expressions in (21) and (22), changing the order of integration, and setting

$$\int_0^\infty f_k(\alpha)\, e^{\alpha-1} e^{\alpha z} d\alpha = \Phi_k(s, z) \quad (k = 1, 2), \tag{24}$$

we obtain /183

$$\left.\begin{aligned} F_k(\rho, z) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Phi_k(s, z) \frac{2^{-s}\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}s\right)} \rho^{s-1} ds; \\ \frac{\partial F_k(\rho, z)}{\partial z} &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Phi_k(s, z) \frac{2^{1-s}\Gamma\left(1-\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}s\right)} \rho^{s-2} ds; \end{aligned}\right\} (k = 1, 2). \tag{25}$$

Let us introduce the functions $U_1(x, z)$ and $U_2(x, z)$ which are harmonic in the $z < 0$-plane, and which are antisymmetrical with respect to the Ox-axis

$$U_k(x, z) = \int_0^\infty \frac{1}{\alpha} f_k(\alpha) \sin(\alpha x)\, e^{\alpha z} d\alpha \quad (k = 1, 2). \tag{26}$$

Differentiating (26) with respect to z, we obtain

$$\frac{\partial U_k(x, z)}{\partial z} \int_0^\infty f_k(\alpha) \sin(\alpha x)\, e^{\alpha z} d\alpha \quad (k = 1, 2). \tag{27}$$

Substituting the following

$$\begin{aligned} \frac{1}{\alpha}\sin(\alpha x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sqrt{\pi} \frac{2^{1-s}\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)}{\Gamma\left(1+\frac{1}{1}s\right)} x^s\alpha^{s-1} ds; \\ \sin(\alpha x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sqrt{\pi} \frac{2^{s-s}\Gamma\left(1+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} x^{s-1}\alpha^{s-1} ds, \end{aligned} \tag{28}$$

in (26) and (27), instead of the functions $\frac{1}{\alpha}$ sin (αx) and sin (αx), changing the order of integration, and employing (24), we obtain

$$\left.\begin{aligned} U_k(x, z) &= \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \Phi_k(s, z)\frac{\sqrt{\pi}\,2^{1-s}\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)}{\Gamma\left(1-\frac{1}{2}s\right)}x^s ds;\\ \frac{\partial U_k(x, z)}{\partial z} &= \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \Phi_k(s, z)\frac{\sqrt{\pi}\,2^{-s}\Gamma\left(1-\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}s\right)}x^{s-1} ds \end{aligned}\right\}(k=1, 2). \tag{29}$$

Employing the formulas /184

$$\begin{aligned} \int\limits_0^x \rho^{2\alpha-1}(x^2-\rho^2)^{\beta-1}d\rho &= \frac{1}{2}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}x^{2\alpha+2\beta-2};\\ \int\limits_x^\infty \rho^{-2\alpha-2\beta-1}(\rho^2-x^2)^{\beta-1}d\rho &= \frac{1}{2}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}x^{-2\alpha}, \end{aligned} \tag{30}$$

we obtain the following from relationships (25) and (29)

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\int\limits_0^x F_k(\rho, z)\frac{\rho d\rho}{\sqrt{x^2-\rho^2}} = \frac{1}{4}\cdot\frac{\partial U_k(x, z)}{\partial x};\\ &\frac{\partial}{\partial x}\int\limits_x^\infty F_k(\rho, z)\frac{\rho d\rho}{\sqrt{\rho^2-x^2}} = \frac{1}{4}\cdot\frac{\partial U_k(x, z)}{\partial z};\\ &\frac{1}{2\pi}\int\limits_0^\rho \frac{\partial U_k(x, z)}{\partial x}\cdot\frac{dx}{\sqrt{x^2-\rho^2}}\; F_k(\rho, z);\\ &\frac{1}{2\pi\rho}\frac{\partial}{\partial\rho}\int\limits_0^\rho \frac{\partial U_k(x, z)}{\partial z}\cdot\frac{xdx}{\sqrt{\rho^2-x^2}} = \frac{\partial F_k(\rho, z)}{\partial z};\\ &\frac{1}{2\pi}\int\limits_\rho^\infty \frac{\partial U_k(x, z)}{\partial x}\cdot\frac{dx}{\sqrt{x^2-\rho^2}} = F_k(\rho, z);\\ &-\frac{1}{2\pi\rho}\cdot\frac{\partial}{\partial\rho}\int\limits_0^\infty \frac{\partial U_k(x, z)}{\partial x}\cdot\frac{xdx}{\sqrt{\rho^2-x^2}} = \frac{\partial F_k(\rho, z)}{\partial z} \end{aligned}\right\}(k=1, 2) \tag{31}$$

On the basis of (31) which are valid in the case z = 0, conditions (19) may be given in the following form

$$\begin{aligned} &\left[\frac{\partial U_1(x, z)}{\partial x}\right]_{z=0} = 4C;\quad \left[\frac{\partial U_2(x, z)}{\partial z}\right]_{z=0} = 4C_1 x;\quad |x|<a;\\ &\left[\frac{\partial U_1(x, z)}{\partial x} - B\frac{\partial U_2(x, z)}{\partial x}\right]_{z=0} = 0;\quad \left[\frac{\partial U_1(x, z)}{\partial z} - A\frac{\partial U_2(x, z)}{\partial z}\right]_{z=0} = 0;\quad |x|>a. \end{aligned} \tag{32}$$

Solution of the plane problem. Criterion of disturbance. Following the procedure of Muskhelishvili (Ref. 3), let us introduce the following notation: S^- -- lower halfplane ($z < 0$); S^+ -- upper halfplane ($z > 0$); L' -- segment $-a < x < a$ of the axis Ox; L'' -- remaining portion of this axis.

The functions $U_1(x, z)$ and $U_2(x, z)$ which are harmonic in the halfplane $z < 0$ will be regarded as real parts of the functions $\Psi_1(\zeta)$ and $\Psi_2(\zeta)$ which are analytical in this halfplane, i.e., /185

$$U_k(x, z) = \frac{1}{2}\Psi_k(\zeta) + \frac{1}{2}\overline{\Psi_k(\zeta)} \quad (\zeta = x + iz; \quad k = 1, 2). \tag{33}$$

Let us introduce the functions $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$ which are analytical over the entire plane ζ, with the exception of the section coinciding with L':

$$\begin{aligned}&\Psi_1'(\zeta) - B\Psi_2'(\zeta) = \Omega_1(\zeta); \quad \Psi_1'(\zeta) - A\Psi_2'(\zeta) = \Omega_2(\zeta) \text{ in } S^-;\\ &\overline{\Psi_1'(\zeta)} - B\overline{\Psi_2'(\zeta)} = -\Omega_1(\zeta); \quad \overline{\Psi_1'(\zeta)} - A\overline{\Psi_2'(\zeta)} = \Omega_2(\zeta) \text{ in } S^+.\end{aligned} \tag{34}$$

The boundary conditions for $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$ on L' are as follows:

$$\left.\begin{aligned}&\frac{A}{A-B}\Omega_1^- - \frac{B}{A-B}\Omega_2^- - \frac{A}{A-B}\Omega_1^+ - \frac{B}{A-B}\Omega_2^+ = 8C;\\ &\frac{1}{A-B}\Omega_1^- - \frac{1}{A-B}\Omega_2^- - \frac{1}{A-B}\Omega_1^+ - \frac{1}{A-B}\Omega_2^+ = -8C_1x\end{aligned}\right\} |x| < a. \tag{35}$$

The conditions on L'' are satisfied by the appropriate selection of the functions Ω_1 and Ω_2. Equations (35) may be reduced to linear conjugate problems, whose solution will be

$$\begin{aligned}\Omega_1(\zeta) = 2(A-B)C_1 i\Big\{a\gamma\Big[\Big(\frac{\zeta-a}{\zeta+a}\Big)^{\gamma} - \Big(\frac{\zeta-a}{\zeta+a}\Big)^{-\gamma}\Big] +\\ +\zeta\Big[\Big(\frac{\zeta-a}{\zeta+a}\Big)^{\gamma} + \Big(\frac{\zeta-a}{\zeta+a}\Big)^{-\gamma}\Big] - 2\zeta\Big\};\end{aligned} \tag{36}$$

$$\begin{aligned}\Omega_2(\zeta) = \frac{2(A-B)}{\sqrt{AB}}C_1 i\Big\{\zeta\Big[\Big(\frac{\zeta-a}{\zeta+a}\Big)^{-\gamma} - \Big(\frac{\zeta-a}{\zeta+a}\Big)^{\gamma} -\\ -a\gamma\Big[\Big(\frac{\zeta-a}{\zeta+a}\Big)^{\gamma} + \Big(\frac{\zeta-a}{\zeta+a}\Big)^{-\gamma}\Big] - 2a\gamma\Big\},\end{aligned} \tag{37}$$

where

$$\gamma = \frac{1}{2\pi i}\ln\frac{A+\sqrt{AB}}{A-\sqrt{AB}}.$$

The boundary values of the functions $\frac{\partial U_k(x, z)}{\partial x}$ and $\frac{\partial U_k(x, z)}{\partial z}$ may be found from the formulas

$$\frac{\partial U_1}{\partial x} = \frac{1}{2(A-B)}\Big\{A[\Omega_1^-(x) - \Omega_1^+(x)] - B[\Omega_2^-(x) + \Omega_2^+(x)]\Big\}; \tag{38a}$$

$$\frac{\partial U_2}{\partial x} = \frac{1}{2(A-B)}\Big\{[\Omega_1^-(x) - \Omega_1^+(x)] - [\Omega_2^-(x) + \Omega_2^+(x)]\Big\}; \tag{38b}$$

$$\frac{\partial U_1}{\partial z} = \frac{i}{2(A-B)}\Big\{A[\Omega_1^-(x) + \Omega_1^+(x)] - B[\Omega_2^-(x) - \Omega_2^+(x)]\Big\}; \tag{38c}$$

$$\frac{\partial U_2}{\partial z} = \frac{i}{2(A-B)}\Big\{[\Omega_1^-(x) + \Omega_1^+(x)] - [\Omega_2^-(x) - \Omega_2^-(x)]\Big\}. \tag{38d}$$

The presence of a crack having the radius a in the body lowers its potential energy by the amount /186

$$W = \frac{1}{2}\iint_{\sigma} p\,(w^{+} - w^{-})\,d\sigma. \tag{39}$$

Here the integration region σ is a circle of the radius a. The surface energy of the crack is

$$U = 2\pi a^{2} T. \tag{40}$$

Taking the fact into account that $w^{+}(x, y, 0) = \phi_3^{+}(x, y, 0)$; $w^{-}(x, y, 0) = \phi_3^{-}(x, y, 0)$, and employing relationships (36), (38a) - (38d), (31), (18) and 15), we obtain

$$W = \frac{2\pi p^{2}}{8} \cdot \frac{\mu_1^{2}\chi_2 + \mu_2^{2}\chi_1 + \mu_1\mu_2(1 + \chi_1\chi_2)}{\mu_1\mu_2[\mu_1(\chi_2 - 1) - \mu_2(\chi_1 - 1)]}(\Theta^{2} + 1)\,\Theta a^{3}, \tag{41}$$

where

$$\chi_1 = \frac{\lambda_1 + 3\mu_1}{\lambda_1 + \mu_1}; \quad \chi_2 = \frac{\lambda_2 + 3\mu_2}{\lambda_2 + \mu_2}; \quad \Theta = \frac{1}{2\pi}\ln\alpha; \tag{42}$$

where $\alpha = \dfrac{\dfrac{\chi_1}{\mu_1} + \dfrac{1}{\mu_2}}{\dfrac{\chi_2}{\mu_2} + \dfrac{1}{\mu_1}}$ — is the bielastic constant.

Substituting the values of W from (41) and of U from (40) in relationship (1), we obtain the magnitude of the disturbing stress as a function of the crack radius

$$p_0 = \sqrt{\frac{2T\mu_1\mu_2[\mu_1(\chi_2 - 1) - \mu_2(\chi_1 - 1)]}{a[\mu_2^{2}\chi_1 + \mu_1^{2}\chi_2 + \mu_1\mu_2(1 + \chi_1\chi_2)](\Theta^{2} + 1)\,\Theta}}. \tag{43}$$

In the case of a homogeneous body ($\mu_1 = \mu_2 = \mu$, $\chi_1 = \chi_2 = \chi$), we obtain

$$p_0 = \sqrt{\frac{4\pi T}{a}} \cdot \frac{\mu}{\chi + 1}. \tag{44}$$

Taking the fact into account that

$$\mu = \frac{E}{2(1 + \nu)} \text{ and } \chi = 3 - 4\nu,$$

we have the following from relationship (44)

$$p_0 = \sqrt{\frac{\pi E T}{2(1 - \nu^{2})a}}, \tag{45}$$

which coincides with the Sneddon result.

In the special case when one of the halfspaces is absolutely rigid ($\mu_1 = \infty$), disturbance occurs in the case

$$p_0 = \sqrt{\frac{16\pi^{3} T\mu^{2}(\chi_2 - 1)}{a\chi_2(\ln^{2}\chi_2 + 4\pi^{2})\ln\chi_2}}. \tag{46}$$

In conclusion, we would like to present the formulas we obtained for determining the normal and shearing stresses outside the crack on the division plane: /187

$$\sigma_z = \frac{4p}{\pi} \cdot \frac{E_1(1-\nu_2^2)+E_2(1-\nu_1^2)}{E_2(1+\nu_1)(1-2\nu_1)-E_1(1+\nu_2)(1-2\nu_2)} \int_1^0 \left[\left(\frac{2}{t} - \right.\right.$$
$$\left. - \frac{2a^2\theta^2}{\rho^2-a^2t^2}\right) \sin\left(\theta \ln \frac{\rho-at}{\rho+at}\right) +$$
$$\left. + \left(\frac{2\rho}{\rho^2-a^2t^2} + \frac{1}{\rho}\right) a\theta \cos\left(\theta \ln \frac{\rho-at}{\rho+at}\right) + \frac{a\theta}{\rho}\right] \frac{dt}{t^2\sqrt{1-t^2}};$$
$$\tau_{\rho z} = \frac{4p}{\pi} \cdot \frac{E_1(1-\nu_2^2)+E_2(1-\nu_1^2)}{E_2(1+\nu_1)(1-2\nu_1)-E_1(1+\nu_2)(1-2\nu_2)} \int_1^0 \left[\left(\frac{2a\theta^2 t}{\rho^2-a^2t^2} - \right.\right. \tag{47}$$
$$\left. - \frac{1}{t}\right) \cos\left(\theta \ln \frac{\rho-at}{\rho+at}\right) +$$
$$\left. + \frac{2\rho a\theta}{\rho^2-a^2t^2} \sin\left(\theta \ln \frac{\rho-at}{\rho+at}\right) + \frac{1}{t}\right] \frac{dt}{t\sqrt{1-t^2}}.$$

The integrals contained in (47) must be obtained by numerical methods.

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STRESS STATE OF A PLANE WEAKENED BY A BROKEN-LINE CRACK /188

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When an infinite plate which is weakened by a crack is subjected to tension by forces which are not perpendicular to the crack axis, as experiments have shown, the crack develops at an angle to the initial direction. It is of interest to determine the critical load in the case of such a crack propagation. The concepts of Griffith (Ref. 4) must be utilized for this purpose. However, we must know the stress distribution in a plane weakened by a broken-line crack in order to solve the problem of the crack development.

Formulation of the problem. An infinite plane weakened by a broken-line crack; the rupture angle ϕ is so small that $\sin 2\phi = 2\phi$, $\cos 2\phi = 1$; the plane is subjected to tension by stresses applied at infinity having the intensity p; these stresses influence the middle section of the crack at the angle β; the length of the inclined sections is small (the beginning of the crack development is examined). It is assumed that the opposite edges of the crack lag behind over its entire length.

The boundary conditions in the middle of the crack have the following form (the notation corresponds to that given in [Ref. 2])

$$Y_y^+ = Y_y^- = X_y^+ = X_y^- = 0.$$

We shall introduce a local coordinate system which is oriented along the crack in the inclined sections (Figure 1). In this system, we have

$$Y_{y'}^{\prime +} = Y_{y'}^{\prime -} = X_{y'}^{\prime +} = X_{y'}^{\prime -} = 0.$$

We may express the boundary conditions in the inclined sections in terms of the stress components in the unprimed axes based on the well known formulas of the elasticity theory. Taking the smallness of the angle ϕ into account, we obtain

$$Y'_{y'} = Y_y - 2\varphi X_y; \quad X'_{y'} = (Y_y - X_x)\varphi + X_y.$$

Since the angle ϕ is small and the length of the inclined sections is also /189
small, we shall assume that the stresses $Y_{y'}^{\prime +}$, $Y_{y'}^{\prime -}$, $X_{y'}^{\prime +}$, $X_{y'}^{\prime -}$ have no influence at the points of the crack inclined sections, but do have an influence at the points on the abscissa axis located under them. Such a procedure is generally accepted, for example, in the theory of a thin wing. We thus arrive at the problem of the tension of a plane containing a rectilinear crack with different boundary conditions at different sections of its edge (Figure 2):

$$\begin{gathered} Y_y^+ = Y_y^- = X_y^+ = X_y^- = 0 \text{ for } -a < t < a, \\ \left.\begin{aligned} &Y_y^+ = 2\varphi X_y^+ = 0, \quad Y_y^- - 2\varphi X_y^- = 0 \\ &(Y_y^+ - X_x^+)\varphi + X_y^+ = 0, \quad (Y_y^- - X_x^-)\varphi + X_y^- = 0 \end{aligned}\right\} \begin{aligned} &-a < t < -\alpha; \\ &\alpha < t < a. \end{aligned} \end{gathered} \tag{1}$$

Let us employ the formulas of N. I. Muskhelishvili (Ref. 2) expressing the stress components in terms of two piecewise analytical functions $\Phi(z)$ and $\Omega(z)$ of a complex variable

$$Y_y - iX_y = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}; \tag{2a}$$

$$Y_y - X_x + 2iX_y = 2[\bar{z}\Phi'(z) + \Psi(z)]; \tag{2b}$$

$$Y_y + X_x = 4\,\mathrm{Re}\,\Phi(z); \tag{2c}$$

$$\Psi(z) = \bar{\Omega}(z) - \Phi(z) - z\Phi'(z). \tag{2d}$$

For large $|z|$, we have

$$\begin{aligned} \Phi(z) &= \Gamma - \frac{X+iY}{2\pi(\varkappa+1)} \cdot \frac{1}{z} + 0\left(\frac{2}{z^2}\right); \\ \Omega(z) &= \bar{\Gamma} + \bar{\Gamma}' + \frac{\varkappa(X-iY)}{2\pi(\varkappa+1)} \cdot \frac{1}{z} + 0\left(\frac{1}{z^2}\right), \end{aligned} \tag{3}$$

where

$$\Gamma = B + iC; \quad \Gamma' = B' + iC';$$

$$B = -\frac{p}{4}; \quad C = 0; \quad B' = \frac{p}{2}\cos 2\beta; \quad C' = -\frac{p}{2}\sin 2\beta.$$

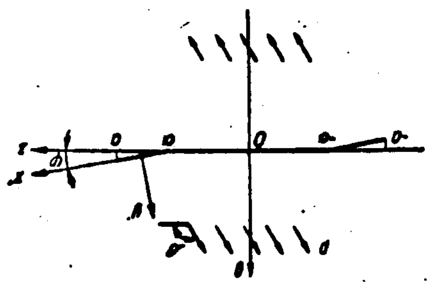

Figure 1

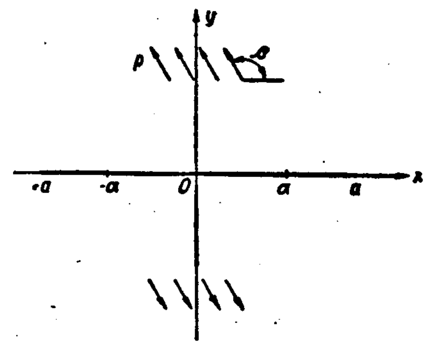

Figure 2

Employing formulas (2) and boundary conditions (1), after simple transformations we arrive at the following boundary value problems for the functions of the complex variable: /190

$$\left.\begin{aligned} [\Phi - \bar{\Phi} - \Omega + \bar{\Omega}]^+ &= [\Phi - \bar{\Phi} - \Omega + \bar{\Omega}]^-; \\ [\Phi + \bar{\Phi} + \Omega + \bar{\Omega}]^+ &= -[\Phi + \bar{\Phi} + \Omega + \bar{\Omega}]^-; \\ [\Phi + \bar{\Phi} - \Omega - \bar{\Omega}]^+ &= [\Phi + \bar{\Phi} - \Omega - \bar{\Omega}]^-; \\ [\Phi - \bar{\Phi} + \Omega - \bar{\Omega}]^+ &= -[\Phi - \bar{\Phi} + \Omega - \bar{\Omega}]^-; \end{aligned}\right\} -a < t < a; \tag{4}$$

$$\left.\begin{aligned} [k\Phi - \bar{\Phi} - k\Omega + \bar{\Omega}]^+ &= [k\Phi - \bar{\Phi} - k\Omega + \bar{\Omega}]^-; \\ [k\Phi + \bar{\Phi} + k\Omega + \bar{\Omega}]^+ &= -[k\Phi + \bar{\Phi} + k\Omega + \bar{\Omega}]^-; \\ [\Phi + k\bar{\Phi} - k\Omega - \bar{\Omega}]^+ &= [\Phi + k\bar{\Phi} - k\Omega - \bar{\Omega}]^-; \\ [\Phi - k\bar{\Phi} + k\Omega - \bar{\Omega}]^+ &= -[\Phi - k\bar{\Phi} + k\Omega - \bar{\Omega}]^-; \end{aligned}\right\} \begin{aligned} &-a < t < -\alpha; \\ &\alpha < t < a. \end{aligned} \tag{5}$$

Here $k = \frac{1 - 2i\phi}{1 + 2i\phi}$. It may be readily seen that the first two relationships (5) are linear combinations of the relationships (4) and, consequently, are valid over the entire length of the section (- a, a).

Solving these problems of linear conjugation, we obtain

$$\begin{aligned} \Phi(z) &= -k\bar{\Omega}(z) + \frac{i(C' + 2\varphi B')}{1 - 2i\varphi} + \frac{(2B + B' - 2\varphi C')z}{(1 - 2i\varphi)\sqrt{z^2 - a^2}}; \\ \Omega(z) &= -\bar{k}\bar{\Phi}(z) - \frac{i(C' + 2\varphi B')}{1 - 2i\varphi} + \frac{(2B + B' - 2\varphi C')z}{(1 - 2i\varphi)\sqrt{z^2 - a^2}}. \end{aligned} \tag{6}$$

We must solve two more linear conjugate problems:

$$\left.\begin{array}{l}[\Phi-\bar{\Omega}+k\bar{\Phi}-k\bar{\Omega}]^{+}=\quad[\Phi-\bar{\Omega}+k\bar{\Phi}-k\Omega]^{-};\\ [\Phi-\bar{\Omega}-k\bar{\Phi}+k\Omega]^{+}=-[\Phi-\bar{\Omega}-k\bar{\Phi}+k\Omega]^{-}\end{array}\right\}\begin{array}{l}-a<t<-\alpha;\\ \alpha<t<a.\end{array} \qquad (7)$$

Combining and subtracting these equations, as well as the last two equations (4), we shall have the following relationships:

$$\left.\begin{array}{l}[\Phi-\bar{\Omega}]^{+}=k[\bar{\Phi}-\Omega]^{-};\\ [\Phi-\Omega]^{-}=k[\bar{\Phi}-\Omega]^{+}\end{array}\right\}\begin{array}{l}-a<t<-\alpha;\\ \alpha<t<a;\end{array}$$

$$\left.\begin{array}{l}[\Phi-\bar{\Omega}]^{+}=\quad[\bar{\Phi}-\Omega]^{-};\\ [\Phi-\bar{\Omega}]^{-}=\quad[\bar{\Phi}-\Omega]^{-}\end{array}\right\}-\alpha<t<\alpha.$$

Let us employ the following notation

$$\begin{array}{l}\Phi(z)-\bar{\Omega}(z)=F_1(z);\\ \bar{\Phi}(z)-\Omega(z)=F_2(z).\end{array} \qquad (8)$$

We then obtain the following relationships on the crack edge:

$$\left.\begin{array}{l}F_1^{+}(t)=kF_2^{-}(t);\\ F_1^{-}(t)=kF_2^{+}(t)\end{array}\right\}\begin{array}{l}-a<t<-\alpha;\\ \alpha<t<a;\end{array}$$

$$\left.\begin{array}{l}F_1^{+}(t)=F_2^{-}(t);\\ F_1^{-}(t)=F_2^{+}(t);\end{array}\right\}-\alpha<t<\alpha. \qquad (9)$$

Let us map the exterior of the segment (- a, a) onto the exterior of a circle having the unit radius (Figure 3) by means of the function /191

$$z=\omega(\zeta)=\frac{a}{2}\left(\zeta+\frac{2}{\zeta}\right).$$

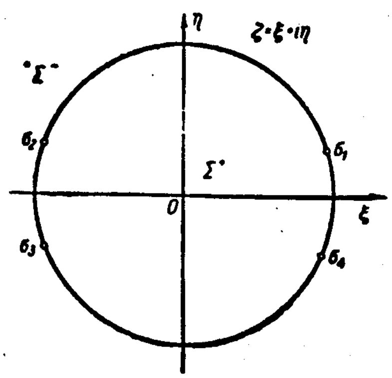

Figure 3

The following points

$$\sigma_1=\frac{\alpha+i\sqrt{a^2-\alpha^2}}{a};$$

$$\sigma_2=\frac{-\alpha+i\sqrt{a^2-\alpha^2}}{a}.$$

correspond to the points α, $-\alpha$, located on the upper edge of the section.

The following points

$$\sigma_3=\frac{-\alpha-i\sqrt{a^2-\alpha^2}}{a};$$

$$\sigma_4=\frac{\alpha-i\sqrt{a^2-\alpha^2}}{a}.$$

correspond the points $-\alpha$, α, located on the lower edge of the section.

We shall employ Σ^+ to designate the interior of the circle, and shall employ Σ^- to designate the exterior. The mapping changes $F_i^+(t)$ $(i = 1, 2)$ into $F_i^-(\sigma)$, and $F_i^-(t)$ into $F_i^-(\bar{\sigma})$.

Employing the notation

$$\sigma_1\sigma_2 + \sigma_3\sigma_4 = L' \text{ and } \sigma_4\sigma_1 + \sigma_2\sigma_3 + L'',$$

we may write relationships (9) in the mapped region in the following form:

$$F_1^-(\sigma) = kF_2^-(\bar{\sigma}) \text{ on } L'';$$
$$F_1^-(\sigma) = F_2^-(\bar{\sigma}) \text{ on } L'.$$

The function $F_1(\zeta)$ is not determined within the circle. Let us set $F_1(\zeta) = F_2\left(\frac{1}{\zeta}\right)$ in the case $|\zeta| < 1$.

Extending the function $F_1(\zeta)$ over the entire plane of the variable ζ, we obtain the linear conjugate problem for it:

$$F_1^+ = kF_1^- \text{ on } L'';$$
$$F_1^+ = F_1^- \text{ on } L'.$$

The solution of this problem may be written in the following form

$$F_1(\zeta) = \left[\frac{(\zeta-\sigma_1)(\zeta-\sigma_3)}{(\zeta-\sigma_2)(\zeta-\sigma_4)}\right]^{\gamma} \frac{P_n(\zeta)}{\zeta^2-1}.$$

Here $\gamma = \frac{1}{2\pi i} \ln k$; $P_n(\zeta) = A_1\zeta^2 + A_2\zeta + A_3$. In order to determine the coefficients A_1, A_2 and A_3, we have the conditions: /192

(1) In the case $\zeta \to \infty$ $F_1(\zeta) \to -\Gamma'$;

(2) In the case $\zeta \to 0$ $F_1(\zeta) \to -\Gamma'$;

(3) In the Laurent expansion of the function $F_1(\zeta)$ in the vicinity of an infinitely removed point without the term ζ^{-1}, as follows from (3), since in our case $X = Y = 0$.

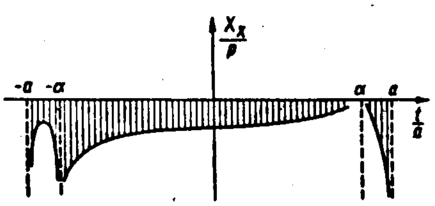

Figure 4

We obtain the following from these conditions

$$A_1 = -\Gamma'; \; A_2 = 0;$$
$$A_3 = \left(\frac{\sigma_2\sigma_4}{\sigma_1\sigma_3}\right)^{\gamma} \bar{\Gamma}' = m\bar{\Gamma}'.$$

Finally, we have

$$F_1(\zeta) = -\left[\frac{(\zeta-\sigma_1)(\zeta-\sigma_3)}{(\zeta-\sigma_2)(\zeta-\sigma_4)}\right]^{\gamma} \frac{\Gamma'\zeta^2 - m\bar{\Gamma}'}{\zeta^2-1}.$$

The functions $\Phi(z)$ and $\Omega(z)$ are found from (8) and (6):

$$\Phi(z) = \frac{1}{2}\left[-(1+2i\varphi)F_1(z) + i(C' + 2\varphi B') + \frac{(2B + B' - 2\varphi C')z}{\sqrt{z^2-a^2}}\right];$$
$$\Omega(z) = \frac{1}{2}\left[(1+2i\varphi)F_2(z) - i(C' + 2\varphi B') + \frac{(2B + B' - 2\varphi C')z}{\sqrt{z^2-a^2}}\right].$$

Utilizing formulas (2), we may compute the stress at the crack edge. The graph showing the stress X_x^+ is shown in Figure 4. Thus, we have completely solved the problem of the stress state in a plane weakened by a broken-line crack.

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IM

PROPAGATION OF CRACKS OF A NEARLY CIRCULAR PLANAR FORM /193

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Employing the results given in (Ref. 2), this article develops an approximate method for determining the magnitude of limiting and breaking load for an unlimited brittle body weakened by a plane isolated crack having a nearly circular planar form, when the body is loaded by a monotonically increasing (proportional to a certain parameter) system of external stresses which are symmetrical with respect to the crack plane.

Formulation of the problem. Let us investigate an unlimited brittle body with an isolated internal crack. We shall assume that this body is referred to a system of Cartesian rectangular coordinates xyz so that the plane xOy(z = = 0) coincides with the crack plane, and the crack occupies a certain limited area S_0 in this plane. Around the region S_0 in the z = 0 plane, let us depict a circle with the radius a. We shall assume that the origin of the system xyz is located in the center of the circle. In addition, we shall employ $R_0(\beta)$ to designate the radius vector of the profile L_0 of the boundary for the region S_0, where β is the polar angle shown in the figure.

It is assumed that the crack S_0 bounded by the profile L_0 has an almost circular form if the maximum value of the function:

$$\varepsilon(\beta) = a - R_0(\beta) \tag{1}$$

is small as compared with the radius a of the circle. The function $\varepsilon(\beta)$ represents a non-negative, limited, and periodic function with the period $2\pi (0 \leq \leq \beta \leq 2\pi)$.

Let the brittle body containing an internal planar crack, having a planar in the crack plane having an almost circular planar (in the crack plane) form be subjected to tension by monotonically increasing stresses Q, directed /194
symmetrically with respect to the crack plane. In particular, we shall assume that $Q = \sigma_z(x, y, \infty) = \sigma_\infty$.

For this case, we shall determine the smallest stress $Q = Q_*$ at which the crack is in a state of dynamic equilibrium (Ref. 1) at any one point of its profile, i.e., we shall determine the smallest load $Q = Q_*$ at which the crack under consideration begins to be propagated over the body cross-section. The load Q_* is called a critical, or limiting, load. However, when the external load value $Q = Q_*$ is reached, this does not always lead to unstable crack propagation over the body cross-section and, consequently, the load $Q = Q_*$ is not always a breaking load.

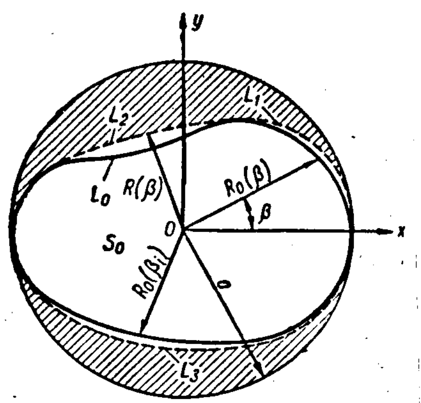
A determination of the load $Q = Q_*$ at which unstable crack propagation sets in and at which the body is destroyed, is also of importance in determining the strength properties of solid bodies weakened by cracks.

On the basis of results given in (Ref. 2), a method is given below for the approximate determination of the load Q_* and Q_{**} for the case when a body is weakened by a plane isolated crack having an almost circular planar form, and the external load Q is symmetrical with respect to the crack plane.

Main equations of the problem. The intensity of cohesion in the vicinity of the profile L_0 of this crack also increases monotonically during the monotonic increase in the external load Q which is applied to a brittle body with a plane crack S_0 (here the plane of the drawing is the crack plane, see the figure). As is known form (Ref. 1, 2), the following conditions

$$\lim_{s_i \to 0} \sqrt{s_i}\,\sigma_z^*(s_i,\ \beta_i,\ 0) = \frac{K}{\pi}\,(i = 1,\ 2,\ 3,\ \ldots), \tag{2}$$

are satisfied for the load $Q = Q_*$ in the vicinity of certain points $R_0(\beta_i)$ of the profile L_0, where K is the cohesion modulus (Ref. 1); $\sigma_z^*(S_i,\ \beta_i,\ 0)$ -- elastic tensile stresses caused by the load $Q = Q_*$ in the vicinity of the points $R_0(\beta_i)$; S_i -- distance between the points of the body located in the crack plane and the crack profile.



Thus, the problem of determining the limiting load $Q = Q_*$ for a brittle body with a plane crack having a curvilinear profile may be reduced to determining the elastic stresses in the vicinity of the crack profile for a given external load -- i.e., to determining the stress concentration in the body with the crack. For this reason, the study of stress concentration around cuts (cracks, narrow cavities) in a deformed elastic body is of significant importance for the theory of crack propagation during the brittle fracture of solid bodies. /195

However, the solution of this problem in the general case -- i.e., for an arbitrary profile L_0 -- entails great mathematical difficulties.

The study (Ref. 2) illustrated an approximate method for calculating the stresses $\sigma_z(S_i,\ \beta_i,\ 0)$ in the vicinity of a plane crack profile in an unlimited elastic body, when this profile has an almost circular form and when the body is subjected to tension by a system of external stresses Q which are symmetrical with respect to the crack plane. In accordance with the results given in (Ref. 2), in this case, the stress component $\sigma_z(r,\ \beta,\ 0)$ in the vicinity of the crack profile may be approximately determined (within an accuracy of small values of $\varepsilon(\beta)/a$, inclusively) by the following formula:

$$\sigma_z(r,\ \beta,\ 0) = \frac{1}{\sqrt{r^2 - R_0^2(\beta)}}\Big\{\psi(a,\ \beta) + \frac{1}{2}\varepsilon(\beta)\,\psi_r'(a,\ \beta) + \tag{3}$$

$$+(r-a)\psi_r'(a, \beta)+$$
$$+\frac{1}{4\pi a}\int_0^{2\pi}\frac{d}{d\alpha}\Big[\varepsilon(\alpha)\psi(a, \alpha)\Big]\operatorname{ctg}\frac{\beta-\alpha}{2}d\alpha\Big\}. \tag{3}$$

Here $0 \leq \beta \leq 2\pi$; $R_0(\beta) \leq r \leq a$; $r = s + R_0(\beta)$; $\psi(\alpha, \beta)$ and $\varepsilon(\beta)$, $R_0(\beta)$ are known functions, and $\varepsilon(\beta)$ may be determined by equation (1) and

$$\psi(r, \beta)=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{a}\frac{\sqrt{a^2-\rho^2}\,[t(\rho, \alpha)+p(\rho, \alpha)]\,\rho d\rho d\alpha}{r^2+\rho-2r\rho\cos(\alpha-\beta)}+ \tag{4}$$
$$+\sqrt{r^2-a^2}\,p(r, \beta),$$

where t (ρ, α) is the external pressure applied to the crack walls; p (ρ, α) -- normal stresses $\sigma_z(\rho, \alpha, 0)$ arising in a continuous (without a crack) elastic body in the z = 0 plane as the result of external stresses Q.

If we now employ the formulas (2) and (3), after certain transformations we may readily obtain the approximate equation for computing the limiting values of the load $Q = Q_*^{(i)}$ at which a dynamically balanced state sets in at the points $R_0(\beta_i)$ of the crack profile L_0, i.e., when the crack begins to be propagated over the body cross-section in the vicinity of the points $R_0(\beta_i)$. /196
This equation has the following form:

$$\frac{\pi}{\sqrt{2R_0(\beta_i)}}\Big\{\psi_*(a, \beta_i)-\frac{1}{2}\varepsilon(\beta_i)\psi_{*r}'(a, \beta_i)+ \tag{5}$$
$$+\frac{1}{4\pi a}\int_0^{2\pi}\frac{d}{d\alpha}\Big[\varepsilon(\alpha)\psi_*(a, \alpha)\Big]\cdot\operatorname{ctg}\frac{\beta_i-\alpha}{2}d\alpha\Big\}=K.$$

Here we employ $\psi_*(\alpha, \beta)$ to designate the value of the function $\psi(a, \beta)$ at the load $Q = Q_*$.

In addition, employing equation (5), in each specific case we may determine the load $Q = Q_*$, as is done in (Ref. 2, 3).

The determination of the limiting (critical) load $Q = Q_*$ for a brittle body weakened by a plane crack having an almost circular planar form is only the first stage in solving this problem. In order to solve this problem completely, it is necessary to determine the load $Q = Q_{**}$ also, i.e., the load at which unstable development of the crack sets in, and at which the body is destroyed.

We should note that when the external load Q reaches the value Q_*, the crack profile L_0 in the vicinity of the points $R_0(\beta_i)$ is in a state of dynamic equilibrium. With a further increase (even a small increase) in the load Q, the profile begins to be displaced over the body cross-section in the plane in which the crack is located.

Let us determine the profile L (dashed line) of the dynamically stable crack S which is formed from the crack S_0 (solid line) during the monotonic increase of the load Q up to the value

$$Q = \lambda Q_*, \tag{6}$$

where $\lambda \geqq 1$.

The radius vector of the profile L of the crack S may be represented in the form of the following equation

$$R(\beta) = R_0(\beta) + \varepsilon_1(\beta), \tag{7}$$

where $R_0(\beta)$ is the radius vector of the profile L_0; $\varepsilon_1(\beta)$ -- the still unknown function of the argument β and the parameter λ.

If the function $\varepsilon_1(\beta)$ is determined for the given profile configuration L_0 of the initial crack S_0 and the given value of the parameter λ, the problem of the profile of a dynamically stable crack will also be solved.

We have the following conditions for determining the function $\varepsilon_1(\beta)$. The function $\varepsilon_1(\beta)$ is positive and does not equal zero on the sections L_j ($j = 1, 2, 3, \ldots$) of the profile L which do not coincide with the intitial profile L_0; the function $\varepsilon_1(\beta)$ equals zero on the sections $(L - L_j)$, i.e., on the sections of L which coincide with the profile L_0. In addition, we should note that the arcs L_j ($j = 1, 2, 3, \ldots$) must change smoothely into the profile L_0 of the initial crack. Consequently, the function $\varepsilon_1(\beta)$ must satisfy the following boundary conditions: /197

$$\begin{gathered} \varepsilon_x(\beta^*_{1j}) = \varepsilon_1(\beta^*_{2j}) = 0; \quad j = 1, 2, 3, \ldots; \\ \left[\frac{d\varepsilon_1(\beta)}{d\beta}\right]_{\beta=\beta^*_{1j}} = \left[\frac{d\varepsilon_1(\beta)}{d\beta}\right]_{\beta=\beta^*_{2j}} = 0, \end{gathered} \tag{8}$$

where we employ β^*_{1j} and β^*_{2j} to designate the polar angles corresponding to the initial and final points of the arc L_j.

Since the sections L_j ($j = 1, 2, 3, \ldots$) of the profile L (which do not coincide with the initial profile L_0) are in a state of dynamic equilibrium for the load $Q = \lambda Q_*$, where $\lambda \geqq 1$, this condition (5) must be satisfied for these sections.

Employing equations (1) and (7), and the function $\varepsilon(\beta)$ contained in equation (5), we may write the following:

$$\varepsilon(\beta) = a - R(\beta) = \varepsilon_0(\beta) - \varepsilon_1(\beta), \tag{9}$$

where $\varepsilon_0(\beta) = a - R_0(\beta)$ is a known function; $\beta^*_{1j} \leq \beta \leq \beta^*_{2j}$.

Substituting this equation in (5), we obtain

$$\frac{\pi}{\sqrt{2[R_0(\beta)+\varepsilon_1(\beta)]}}\Big\{\psi_*(a,\ \beta)-\frac{1}{2}\varepsilon_0(\beta)\psi'_{*r}(a,\ \beta)+\frac{1}{2}\varepsilon_1(\beta)\psi'_{*r}(a,\ \beta)+ \\ +\frac{1}{4\pi a}\int_0^{2\pi}\frac{d}{d\alpha}[\varepsilon_0(\alpha)\psi_*(a,\ \alpha)]\operatorname{ctg}\frac{\beta-\alpha}{2}d\alpha- \\ -\frac{1}{4\pi a}\sum_{j=1}^{m}\int_{\beta^*_{1j}}^{\beta^*_{2j}}\frac{d}{d\alpha}[\varepsilon_1(\alpha)\psi_*(a,\ \alpha)]\cdot\operatorname{ctg}\frac{\alpha-\beta}{2}d\alpha\Big\}=K, \tag{10}$$

where $\beta^*_{1j} \leq \beta \leq \beta^*_{2j}$; $j = 1, 2, 3, \ldots$.

Equation (5) and, consequently, equation (10) were compiled in (Ref. 2) within an accuracy of terms whose order of smallness is no greater than $\varepsilon(\beta)/a$, where a is the radius of a circle drawn around the profile of the crack S.

The function $\varepsilon_1(\beta)$ to be determined from equation (10) satisfies the inequality

$$0 \leqslant \varepsilon_1(\beta) \leqslant \varepsilon_0(\beta),$$

i.e., the function $\varepsilon_1(\beta)$ is of the same order of smallness as compared with the quantity a as the function $\varepsilon_0(\beta)$. Therefore, without disturbing the accuracy of equation (10), we may simplify it somewhat if we retain only linear terms with respect to $\varepsilon_1(\beta)$ in this equation. Performing the requisite transformations, we may represent equation (10) within an accuracy of small values of $\varepsilon_1(\beta)/a$, inclusively, in the following form: /198

$$\omega(a,\ \beta)\varepsilon_1(\beta)-\frac{1}{4\pi a}\sum_{j=1}^{m}\int_{\beta^*_{1j}}^{\beta^*_{2j}}\frac{d}{d\alpha}[\varepsilon_1(\alpha)\psi_*(a,\ \alpha)]\cdot\operatorname{ctg}\frac{\beta-\alpha}{2}d\alpha=\Phi(a,\ \lambda,\ \beta). \tag{11}$$

Here we introduce the following notation:

$$\omega(a,\ \beta)=\frac{1}{2}\Big\{\psi'_*(a,\ \beta)-\frac{\psi_*(a,\beta)}{a}+\frac{\varepsilon_0(\beta)\psi'_{*r}(a,\psi)}{2a}- \\ -\frac{1}{4\pi a}\int_0^{2\pi}\frac{d}{d\alpha}[\varepsilon_0(\alpha)\psi_*(a,\ \alpha)]\operatorname{ctg}\frac{\beta-\alpha}{2}d\alpha\Big\}; \tag{12}$$

$$\Phi(a,\ \lambda,\ \beta)=\frac{K}{\pi}\sqrt{2R_0(\beta)}-\psi_*(a,\ \beta)+\frac{1}{2}\varepsilon_0(\beta)\psi'_{*r}(a,\ \beta)- \\ -\frac{1}{4\pi a}\int_0^{2\pi}\frac{d}{d\alpha}[\varepsilon_0(\alpha)\psi_*(a,\ \alpha)]\operatorname{ctg}\frac{\beta-\alpha}{2}d\alpha. \tag{13}$$

Equations (10) and (11) are fundamental equations for determining the form of a dynamically balanced crack for a monotonic increase in the external load λQ_* when $\lambda > 1$. Solving this equation with respect to the function $\varepsilon_1(\beta)$ for a given configuration $R_0(\beta)$ of the initial crack and for a given load process, and employing the boundary conditions (8), we may obtain the final function $\varepsilon_1(\beta)$ and the dependence of the angles β^*_{1j}, β^*_{2j} on the parameter λ.

In addition, employing the expression for the function $\varepsilon_1(\beta)$ and equation (7), we may trace the kinetic propagation of the crack (having an almost circular planar form) as the parameter λ increases, when this parameter satisfies the following condition: $1 \leqq \lambda \leqq \lambda_*$.

Here λ_* is the limiting (largest) value of the parameter characterized by the fact that the development of the dynamically balanced crack is stable for all $\lambda < \lambda_*$, and is unstable for $\lambda \geqq \lambda_*$.

If the external load Q reaches the value $\lambda_* Q_*$, further propagation of the dynamically balanced crack becomes unstable, and the body is destroyed. Thus, the magnitude of the breaking load $Q = Q_{**}$ for a body containing a plane isolated crack, which has an almost circular planar form, may be determined by the equation

$$Q_{**} = \lambda_* Q_*. \tag{14}$$

The value λ_* is the largest value of the parameter λ at which the solution /199 of equation (11) exists and at which the following inequality is satisfied

$$\max \{R_0(\beta) + \varepsilon_1(\beta)\} < a. \tag{15}$$

Thus, by employing equation (11), the boundary conditions (7), and formulas (14), (15), in each specific case we may determine the breaking load for a brittle body weakened by a macroscopic crack which has an almost circular planar (crack plane) form, when the body is subjected to tension by a monotonically increasing external load Q which is symmetrical to the crack plane.

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STUDY OF THE STRESS CONCENTRATION NEAR HOLES IN PLATES DURING BENDING /200

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The articles (Ref. 2, 3, 6) investigated the problems of stress concentration around holes in plates during bending within the framework of classical theory. They are represented most extensively in the monograph by G. N. Savin (Ref. 3).

It was found more recently (Ref. 7) that, when only two boundary conditions (out of three natural boundary conditions) are satisfied, the transverse shearing stresses which should equal zero on the profile of a free hole, but which do not equal zero, significantly increase with a decrease in the hole radius.

It is therefore natural to study these problems by means of the special plate theories (Ref. 1, 5, 8) in which it is possible to satisfy all three boundary conditions on the hole profiles. The first attempt in this direction was made by Reissner (Ref. 8). The problem of stress concentration around holes during bending is studied below on the basis of equations advanced by the theory of plates given in (Ref. 5).

We shall write the homogeneous equations of plate bending which were obtained in (Ref. 1) and, as a special case, were obtained from the more general equations in (Ref. 5) encompassing various variations in the formulation of conditions at the boundary, equidistant planes

$$\left.\begin{aligned} &\Delta w + \frac{\partial \gamma_x}{\partial x} + \frac{\partial \gamma_y}{\partial y} = 0; \\ &\Delta \gamma_x - k^2 \gamma_x = k^2 \frac{\partial w}{\partial x} + \frac{3+2\nu}{2(1-\nu)} \cdot \frac{\partial}{\partial x}(\Delta w); \\ &\Delta \gamma_y - k^2 \gamma_y = k^2 \frac{\partial w}{\partial y} + \frac{3+2\nu}{2(1-\nu)} \cdot \frac{\partial}{\partial y}(\Delta w), \end{aligned}\right\} \tag{1}$$

where $k^2 = \frac{5}{2} h^{-2}$ (2h -- plate thickness).

When the boundary value problems are solved, three boundary conditions on the profile are added to system (1) of the sixth order with respect to the normal bending w and to such clearly defined independent quantities as the angles of rotation of the normal element γ_x and γ_y pertaining to the middle plane. For example, for the edge x = const: /201

(1) In displacements and angles of rotation

$$w = \bar{w}; \quad \gamma_x = \bar{\gamma}_x; \quad \gamma_y = \bar{\gamma}_y;$$

(2) In stresses and moments

$$N_x = \bar{N}; \quad M_x = \bar{M}; \quad H_{xy} = \bar{H}.$$

System (1) may be reduced to the following form (Ref. 5)

$$\begin{aligned} &\Delta\Delta w = 0; \\ &\Delta\varphi - k^2\varphi = 0, \end{aligned} \tag{2}$$

and we have

$$\gamma_x = -\frac{\partial w}{\partial x} - \frac{h^2}{1-\nu}\cdot\frac{\partial}{\partial x}(\Delta w) + \frac{\partial\varphi}{\partial y};$$
$$\gamma_y = -\frac{\partial w}{\partial y} - \frac{h^2}{1-\nu}\cdot\frac{\partial}{\partial y}(\Delta w) - \frac{\partial\varphi}{\partial x}. \qquad (3)$$

The stresses and moments may be determined according to the following formulas (Ref. 5):

$$\left.\begin{aligned}
N_x &= \frac{2Eh}{3(1+\nu)}\left(\gamma_x + \frac{\partial w}{\partial x}\right); \quad N_y = \frac{2Eh}{3(1+\nu)}\left(\gamma_y + \frac{\partial w}{\partial y}\right);\\
M_x &= D\left\{\frac{4}{5}\left(\frac{\partial\gamma_x}{\partial x} + \nu\frac{\partial\gamma_y}{\partial y}\right) - \frac{1}{5}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\right\};\\
M_y &= D\left\{\frac{4}{5}\left(\frac{\partial\gamma_y}{\partial y} + \nu\frac{\partial\gamma_x}{\partial x}\right) - \frac{1}{5}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)\right\};\\
H_{xy} &= \frac{D(1-\nu)}{2}\left\{\frac{4}{5}\left(\frac{\partial\gamma_x}{\partial y} + \frac{\partial\gamma_y}{\partial x}\right) - \frac{2}{5}\frac{\partial^2 w}{\partial x\partial y}\right\},
\end{aligned}\right\} \qquad (4)$$

where $D = \dfrac{2Eh^3}{3(1-\nu^2)}$ is the cylindrical rigidity.

The assumed theory of plate bending (Ref. 5) yields the following expressions for the bending stresses:

$$\left.\begin{aligned}
\sigma_x &= \frac{3M_x}{2h^3}z + \frac{3z}{2h(1-\nu)}\left(\frac{1}{5} - \frac{z^2}{3h^2}\right)\left(\frac{\partial N_x}{\partial x} + \nu\frac{\partial N_y}{\partial y}\right);\\
\sigma_y &= \frac{3M_y}{2h^3}z + \frac{3z}{2h(1-\nu)}\left(\frac{1}{5} - \frac{z^2}{3h^2}\right)\left(\frac{\partial N_y}{\partial y} + \nu\frac{\partial N_x}{\partial x}\right);\\
\tau_{xy} &= \frac{3H_{xy}}{2h^3}z + \frac{3z}{4h}\left(\frac{1}{5} - \frac{z^2}{3h^2}\right)\left(\frac{\partial N_x}{\partial y} + \frac{\partial N_y}{\partial x}\right);\\
\tau_{xz} &= \frac{3N_x}{4h^3}(h^2 - z^2);\\
\tau_{yz} &= \frac{3N_y}{4h^3}(h^2 - z^2).
\end{aligned}\right\} \qquad (5)$$

System (1) assumed simple solutions for several cases of the homogeneous stress state of a finite rectangular plate. Let us write these solutions for two cases of plate loading: /202

(1) Two-sided bending by the moments M_x and M_y over the edges

$$w = \frac{1}{2D(1-\nu^2)}[(\nu M_y - M_x)x^2 + (\nu M_x - M_y)y^2];$$
$$\gamma_x = \frac{M_x - \nu M_y}{D(1-\nu^2)}x; \qquad \gamma_y = \frac{-\nu M_x + M_y}{D(1-\nu^2)}y; \qquad (6)$$

(2) Torsion by the moments H distributed over the plate edges

$$w = \frac{H}{D(\nu-1)}xy;$$
$$\gamma_x = \frac{H}{D(1-\nu)}y; \qquad \gamma_y = \frac{H}{D(1-\nu)}x. \qquad (7)$$

Let us write the corresponding solutions in polar coordinates. Let us first limit ourselves to the case of purely cylindrical bending of a finite rectangular plate by the moments $M_x = M$. We have

$$\left.\begin{aligned}
w &= -\frac{M\rho^2}{4D(1-\nu^2)}[(1-\nu) + (1+\nu)\cos 2\theta];\\
\gamma_\rho &= \frac{M\rho}{2D(1-\nu^2)}[(1-\nu) + (1+\nu)\cos 2\theta];
\end{aligned}\right] \qquad (8)$$

$$\gamma_\theta = -\frac{M\rho}{2D(1-\nu)}\sin 2\theta. \tag{8}$$

For stresses and moments, we obtain

$$\left.\begin{aligned} M_\rho &= \frac{M}{2}(1+\cos 2\theta); \quad M_\theta = \frac{M}{2}(1-\cos 2\theta); \\ H_{\rho\theta} &= -\frac{M}{2}\sin 2\theta; \quad N_\rho = N_\theta = 0. \end{aligned}\right. \tag{9}$$

Formulas (9) coincide with the corresponding expressions obtained in the classical theory of thin plate bending (Ref. 2, 3, 6).

On solution of (8) - (9) let us impose the solution $\tilde{W}$, $\tilde{\gamma}_\rho$, $\tilde{\gamma}_\theta$ of system (1) which would satisfy the following conditions on the profile $\rho=a$:

For the case of a hole with rigid insert:

$$\left.\begin{aligned} \frac{\partial w}{\partial s} + \frac{\partial \tilde{w}}{\partial s} &= 0; \\ \gamma_\rho + \tilde{\gamma}_\rho &= 0; \\ \gamma_\theta + \tilde{\gamma}_\theta &= 0; \end{aligned}\right\} \tag{10}$$

For the case of a free hole:

$$\left.\begin{aligned} N_\rho + \tilde{N}_\rho &= 0; \\ M_\rho + \tilde{M}_\rho &= 0; \\ H_{\rho\theta} + \tilde{H}_{\rho\theta} &= 0. \end{aligned}\right\} \tag{11}$$

In addition, the superposition solution must vanish at infinity (in the case $\rho \to \infty$). /203

We shall try to determine these solutions in the following form (Ref. 4)

$$\begin{aligned} w &= W_0(\rho) + \sum_{m=1}^{\infty} W_m(\rho)\cos m\theta + \sum_{m=1}^{\infty} W'_m(\rho)\sin m\theta; \\ \varphi &= \Phi_0(\rho) + \sum_{m=1}^{\infty} \Phi_m(\rho)\cos m\theta + \sum_{m=1}^{\infty} \Phi'_m(\rho)\sin m\theta, \end{aligned} \tag{12}$$

where $W_m(\rho)$, $W'_m(\rho)$ are determined in the customary manner (Ref. 2).

In order to find the function $\Phi_m(\rho)$, we obtain the equation

$$\Phi''_m + \frac{1}{\rho}\Phi'_m - \left(\frac{m^2}{\rho^2} + k^2\right)\Phi_m = 0,$$

the general solution of which may be written as follows (Ref. 8):

$$\Phi_m(\rho) = C_m J_m(k\rho) + D_m K_m(k\rho). \tag{13}$$

Here $J_m(k\rho)$ and $K_m(k\rho)$ are the Bessel and Macdonald functions of the order m of the argument $k\rho$.

In order to fulfill the damping condition at infinity, we must select the solution which only depends on the function $K_m(k\rho)$ (Ref. 4), i.e.,

$$\Phi_m(\rho) = D_m K_m(k\rho). \tag{13a}$$

With allowance for (8) - (11), we finally obtain the solution to be applied $\tilde{W}$, $\tilde{\gamma}_\rho$, $\tilde{\gamma}_\theta$ in the following form

$$\left.\begin{aligned}\tilde{w} &= C_1 \ln\rho + (C_2\rho^{-2} + C_3)\cos 2\theta;\\ \tilde{\gamma}_\rho &= -C_1\rho^{-1} + 2\rho^{-1}\left[C_2\rho^{-2} - \frac{4h^2}{1-\nu}C_3\rho^{-2} + D_2K_2(k\rho)\right]\cos 2\theta;\\ \tilde{\gamma}_\theta &= -kD_0K_0'(k\rho) + \left[2C_2\rho^{-2} + 2C_3\rho^{-1}\left(1 - \frac{4h^2}{1-\nu}\rho^{-2}\right) - \right.\\ &\left. - kD_2K_2'(k\rho)\right]\sin 2\theta.\end{aligned}\right\} \quad (14)$$

The primes designate the derivatives of the Macdonald function with respect to the argument $k\rho$. The integration constants appearing in expressions (14) may be determined from the boundary conditions (10) or (11).

By way of an example, let us examine the case when an absolutely rigid ring (hole with rigid insert) is sealed into a circular hole of a plate.

Condition (10) yields the following values for the constants $C_{i(i=1,\,2,\,3,)}$, $D_{j(j=0,2)}$:

$$C_1 = \frac{Ma^2}{2D(1+\nu)};\quad C_2 = \frac{Ma^4}{4D(1-\nu)}\left[1 - \frac{2}{1+f\left(\frac{a}{h}\right)}\right];$$

$$C_3 = \frac{Ma^2}{2D(1-\nu)}\,\frac{1}{1+f\left(\frac{a}{h}\right)};\quad D_0 = 0;\quad D_2 = -\frac{8h^2}{a^3k(1-\nu)K_2'(ka)}C_3.$$

Here we have /204

$$f\left(\frac{a}{h}\right) = \frac{4h^2}{a^2(1-\nu)}\left[1 + \frac{2K_2(ka)}{akK_2'(ka)}\right].$$

The following expressions are obtained for the stresses and moments:

$$\begin{aligned}M_\rho = \frac{M}{2}\Bigg\{1 + \frac{1-\nu}{1+\nu}\cdot\frac{a^2}{\rho^2} + \Bigg[1 + \frac{1}{1+f\left(\frac{a}{h}\right)}\cdot\frac{a^2}{\rho^2}\Bigg(\frac{4\nu}{1-\nu} + \frac{96h^2}{5\rho^2(1-\nu)} +\\ + \frac{64h^2}{5a^3k(1-\nu)K_2'(ka)}[K_2(k\rho) - k\rho K_2'(k\rho)]\Bigg) -\\ - \frac{3a^4}{\rho^4}\left(1 - \frac{2}{1+f\left(\frac{a}{h}\right)}\right)\Bigg]\cos 2\theta\Bigg\};\end{aligned}$$

$$\begin{aligned}M_\theta = \frac{M}{2}\Bigg\{1 - \frac{1-\nu}{1+\nu}\cdot\frac{a^2}{\rho^2} + \Bigg[-1 + \frac{1}{1+f\left(\frac{a}{h}\right)}\cdot\frac{a^2}{\rho^2}\Bigg(\frac{4}{1-\nu} - \frac{96h^2}{5\rho^2(1-\nu)} -\\ - \frac{64h^2}{5a^3k(1-\nu)K_2'(ka)}[K_2(k\rho) - k\rho K_2'(k\rho)]\Bigg) +\\ + \frac{3a^4}{\rho^4}\left(1 - \frac{2}{1+f\left(\frac{a}{h}\right)}\right)\Bigg]\cos 2\theta\Bigg\};\end{aligned}$$

$$\begin{aligned}H_{\rho\theta} = \frac{M}{2}\Bigg(-1 - \frac{1}{1+f\left(\frac{a}{h}\right)}\cdot\frac{a^2}{\rho^2}\Bigg\{2 - \frac{96h^2}{5\rho^2(1-\nu)} -\\ - \frac{16h^2}{5a^3k(1-\nu)K_2'(ka)}[k^2\rho^2K_2''(k\rho) - k\rho K_2'(k\rho) + 4K_2(k\rho)]\Bigg\} -\end{aligned}$$

$$-\frac{3a^4}{\rho^4}\left[1-\frac{2}{1+f\left(\frac{a}{h}\right)}\right]\Bigg)\sin 2\theta;$$

$$N_\rho = -\frac{4M}{1-\nu}\cdot\frac{a^2}{\rho^3}\left[1+\frac{2\rho^2 K_2(k\rho)}{a^3 k K_2'(ka)}\right]\frac{\cos 2\theta}{1+f\left(\frac{a}{h}\right)};$$

$$N_\theta = -\frac{4M}{1-\nu}\cdot\frac{a^3}{\rho^3}\left[1-\frac{\rho^3 K_2'(k\rho)}{a^3 K_2'(ka)}\right]\frac{\sin 2\theta}{1+f\left(\frac{a}{h}\right)}.$$

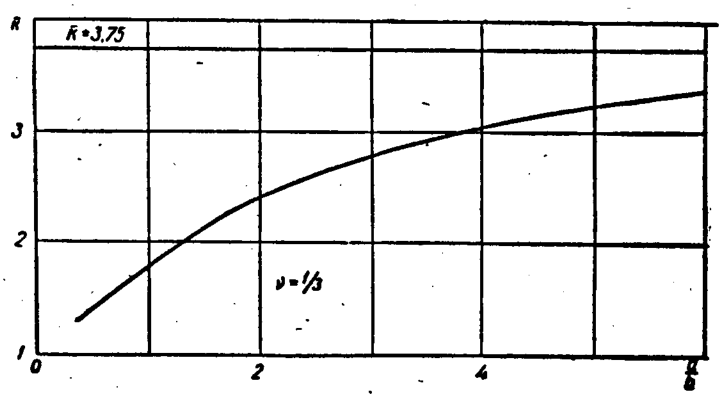
The dependence of all the internal factors obtained on the parameter $\frac{a}{h}$ is apparent.

We may obtain all the previous dependences of classical theory (Ref. 3, 6) in the case $\frac{a}{h} \to \infty$ from the expressions for stresses and moments on the profile of a rigid insert.

The intersection stress N_θ represents an exception. On the profile of a rigid insert N_θ and, consequently, the shearing stresses $\tau_{\theta z}$ equal zero in in contrast to the classical theory, where /205

$$N_\theta = -\frac{4M}{a(1-\nu)}\sin 2\theta.$$

The figure shows the dependence of the concentration coefficient $\bar{k} = \frac{\sigma_\rho^{max}}{\sigma_\rho^0}$ on the parameter $\frac{a}{h}$.



The line corresponding to the concentration coefficient in the customary theory of Kirchhoff ($k = 3.75$ for $\nu = 1/3$) has an asymptotic form for the curve obtained as $\frac{a}{h}$ increases indefinitely.

It may be seen from the graph that, even for holes which exceed the plate thickness by a factor of three, the error of the classical theory when computing the concentration coefficient is 10%. In the case $\frac{a}{h} = 4$, it is 19.2%.

For very small (as compared with the plate thickness) holes, investigations based on special plates theories (Ref. 1, 5, 8) cannot provide reliable results.

Therefore, the graph was not drawn clear to the end, although in the case $\frac{a}{h} \to 0$ a finite limit which is close to unity and which depends on ν is obtained. Such a problem would have to be studied on the basis of the equations of three-dimensional elasticity theory.

For large ratios $\frac{a}{h}$, the calculation of the stress concentration coefficients in classical theory does not lead to significant errors.

The method presented above may be employed to study the nature of stress concentration around holes in several other cases of plate loading. /206

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EFFECT OF FOREIGN MACROINCLUSIONS ON THE DISTRIBUTION OF TEMPERATURE FIELDS AND STRESSES IN ELASTIC BODIES /207

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A study of the stress state in bodies with foreign inclusions, including cavities, is of interest in determining the stability of many structural elements. The results of these studies are also necessary for studying the stability of materials -- in particular, metals -- containing nonuniformities in the form of non-metallic macroinclusions, secondary phases, macrodefects of a different type (pores, cracks, etc.)

Depending upon the conditions under which the material is prepared and processed, these nonuniformities may be three-dimensional, when their dimensions are of the same order in every direction. They may also be surface nonuniformities, if one of the dimensions is small as compared with the others, or they may be linear nonuniformities under the condition that two dimensions are small as compared with the third. In this connection, in the last two cases when a calculational scheme is being selected, it is possible to employ the results derived from the theory of thin plates and shells in the first approximation or, correspondingly, the results derived from the theory of thin rods.

Due to the fact that many important elements of present day construction operate under conditions of nonuniform heating, it is necessary to take into account the temperature stresses caused by the incompatibility of purely thermal deformations when their strength is being determined.

At least two reasons for the occurrence of incompatible thermal deformations in a body may be pointed out. The first reason may be the change in the temperature of the surrounding medium (external heating), which leads to nonstationary, nonuniform heating.

The second, no less important, cause may be found in the cyclic deformation when considerable self-heating occurs, caused by internal energy dissipation (Ref. 11, 18), for sufficiently large loading amplitudes. Nonuniformity of the deformation field in the body, as well as heat exchange over its surface during unfavorable conditions, may produce significant temperature gradients. Thus, in addition to stresses caused by the external force loading, additional temperature stresses may arise which have frequently a decisive influence on the process of fatigue failure (Ref. 19). /208

The material nonuniformity, caused by macroinclusions or cavities, leads in its turn to additional disturbance of the temperature field and stress state. Therefore, the temperature stresses caused by the disturbance of thermal fluxes in the vicinity of foreign inclusions -- particularly cavities and holes -- must also be taken into account when investigating the stability of materials functioning under conditions of nonuniform heating.

It is apparent that the magnitude and law of the temperature stress distribution depends on the nature of the temperature field, which is determined by the solution of the thermoconductivity problem. Therefore, it is natural to investigate the problem of determining temperature stresses together with the corresponding problem of thermoconductivity. A fairly comprehensive account of the initial assumptions, the derivation of the main equations, and also the solution of many specific problems may be found in well known monographs on thermoelasticity and thermocondictivity (Ref. 7-10, 14, 15, 21). In this article, we shall only deal with certain problems related to the influence of elastic nonuniformities on the distribution of temperature fields and temperature stresses in elastic bodies.

The majority of articles published on this subject (Ref. 1, 12, 22-25) pertain to the case of plane deformation, when it is assumed that the temperature t depends *a priori* on two three-dimensional variables (xy), and may be determined by the solution of the differential equation

$$a^2 \Delta t = \frac{\partial t}{\partial \tau}, \quad (1)$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$; a^2 -- temperature conductivity ; τ -- time.

Such a two-dimensional temperature distribution -- which we shall call a plane temperature field from this point on -- is possible in cylindrical bodies, whose end faces are thermally insulated, and the initial and boundary conditions on the cylindrical surface are identical in any transverse cross-section.

The articles mentioned above employ this formulation to investigate the stationary problem of the disturbance of a uniform thermal flux in the vicinity of cylindrical cavities of a different type, whose surfaces are thermally insulated. Since the temperature field is stationary, the determination of temperature stresses may be reduced to determining the stresses from the corresponding dislocations (Ref. 13). /209

It is apparent that in the case of the temperature problem the results pertaining to plane deformation cannot be directly transferred to the case of the generalized plane stress state, as is done in the force problem, if the end faces of the plate are not insulated. This is due to the fact that the heat exchange on these surfaces significantly changes the formulation of, and differential equation of the thermoconductivity problem. In this case, either the three-dimensional thermoconductivity problem is investigated, or a temperature value is introduced into the examination which is averaged over the thickness and which satisfies the following equation in the case of heat exchange which is symmetrical with respect to the middle plane:

$$\Delta t - \varepsilon t = \frac{1}{a^2} \cdot \frac{\partial t}{\partial \tau} - \varepsilon t_c, \quad (2)$$

where $\varepsilon = \frac{\alpha}{\lambda h}$; α -- coefficient of plate heat transfer; λ -- thermoconductivity; 2h -- thickness; t_c -- temperature of the surrounding medium.

In the case $\varepsilon = 0$, the form of equations (1) and (2) coincide. However, since the averaged temperature value occurs in (2), it is advantageous to call the corresponding field the generalized plane temperature field.

In both cases, the stresses may be determined by the following formulas

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}; \quad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2}; \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \tag{3}$$

where U satisfies the inhomogeneous, biharmonic equation

$$\Delta\Delta U = -\alpha_t E\,\Delta t, \tag{4}$$

whose general solution may be represented in the following form

$$U = \mathrm{Re}\,[\bar{z}\varphi(z) + \chi(z)] - \frac{\alpha_t E}{4}\int\int t\,dz\,d\bar{z}, \tag{5}$$

where α_t is the temperature coefficient of linear expansion; E -- Young's modulus.

Thus, in determining the analytical functions $\phi(z)$ and $\psi(z) = \chi'(z)$, in the case of the first or second main problems the following boundary conditions hold

$$\begin{aligned} &\varphi(z) + z\overline{\varphi'(z)} + \psi(z) = \frac{\alpha_t E}{2}\int t\,dz + C; \\ &\varkappa\varphi(z) - z\overline{\varphi'(z)} - \psi(z) = -\frac{\alpha_t E}{2}\int t\,dz. \end{aligned} \tag{6}$$

These conditions, and also the formulas of Kolosov-Muskhelishvili /210

$$\left.\begin{aligned} &2\mu(u+iv) = \varkappa\varphi(z) - \overline{z\varphi'(z)} - \overline{\psi(z)} + \frac{\alpha_t E}{2}\int t\,dz; \\ &\sigma_{xx} + \sigma_{yy} = 2[\varphi'(z) + \overline{\varphi'(z)}] - d_t E t; \\ &\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2[\bar{z}\varphi''(z) + \psi'(z)] - \alpha_t E\int \frac{\partial t}{\partial z}\,d\bar{z}, \end{aligned}\right\} \tag{7}$$

corresponding to the temperature change were first obtained by N. N. Lebedev (Ref. 6, 7).

In the case of plane deformation, we must set $\alpha^* = \alpha(1 + \nu)$, $\nu^* = \frac{\nu}{1 - \nu}$, $E^* = \frac{E}{1 - \nu^2}$ instead of α, ν, E. We may take the following for nonstationary, plane temperature field, instead of (5)

$$U = \mathrm{Re}\,[\bar{z}\varphi(z) + \chi(z)] - a^2\alpha_t E\int_\infty^\tau t\,d\tau,$$

as a result of which (6), (7) may be rewritten in the following form

$$\varphi(z) + z\overline{\varphi'(z)} + \psi(z) = 2a^2\alpha_t E\frac{\partial}{\partial z}\int_\infty^\tau t\,d\tau + C;$$

$$\varkappa\varphi(z) - z\varphi'(z) - \overline{\psi(z)} = -2a^2\alpha_t E\frac{\partial}{\partial z}\int_\infty^\tau t\,d\tau;$$

$$2\mu(u+iv) = \varkappa\varphi(z) - z\varphi'(z) - \overline{\psi(z)} + 2a^2\alpha_t E\frac{\partial}{\partial z}\int_\infty^\tau t\,d\tau;$$

$$\sigma_{xx} + \sigma_{yy} = 2[\varphi'(z) + \overline{\varphi'(z)}] - 4\alpha_t E(t - t|_{\tau=\infty});$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2[\bar{z}\varphi''(z) + \psi'(z)] - 4a^2\alpha_t E\int_{\infty}^{\tau}\frac{\partial^2 t}{\partial \bar{z}^2}d\tau.$$

If the generalized plane temperature field is stationary, and the function t_c is harmonic, we may also write the following instead of (5)

$$U = \mathrm{Re}[\bar{z}\varphi(z) + \chi(z)] - \frac{\alpha_t E}{\varepsilon}t.$$

Thus, relationships (6), (7) will have the following form

$$\varphi(z) + z\overline{\varphi'(z)} + \psi(z) = 2\frac{\alpha_t E}{\varepsilon}\cdot\frac{\partial t}{\partial \bar{z}} + C;$$

$$\varkappa\varphi(z) - z\varphi'(z) - \overline{\psi(z)} = -2\frac{\alpha_t E}{\varepsilon}\cdot\frac{\partial t}{\partial \bar{z}};$$

$$2\mu(u + iv) = \varkappa\varphi(z) - z\varphi'(z) - \overline{\psi(z)} + 2\frac{\alpha_t E}{\varepsilon}\cdot\frac{\partial t}{\partial \bar{z}};$$

$$\sigma_{xx} + \sigma_{yy} = 2[\varphi'(z) + \overline{\varphi'(z)}] - 4\alpha_t Et;$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2[\bar{z}\varphi''(z) + \psi'(z)] - 4\frac{\alpha_t E}{\varepsilon}\cdot\frac{\partial^2 t}{\partial \bar{z}^2}.$$

These relationships enable us to solve the class of plane problems being investigated in the case of nonstationary thermal regimes both for a plane and for a generalized plane temperature field. /211

However, the formulation of the considerations presented above, pertaining to the theoretical and applied value of the problem regarding distribution of temperature fields and stresses in bodies with foreign inclusions, requires further generalization. This generalization may be done by defining the boundary conditions more accurately, particularly the stresses of the thermal and mechanical contact of solid bodies.

The formulation of conditions for an ideal contact assumes that the contiguous bodies are divided by an ideal (mathematically) surface, and each of the bodies is uniform up to the dividing surface. In actuality, a certain transitional layer may be located between the bodies; the properties of this layer may differ from the properties of the contiguous bodies.

In this connection, a formulation of the problem based on the following computational model is advantageous. Assuming that the transitional layer thickness is small as compared with other dimensions, we shall regard it as a thin shell with definite physico-mathematical characteristics. Keeping these characteristics constant and extending the shell thickness to zero, we obtain a certain physical surface dividing the bodies with characteristic, definite values of the physico-mechanical characteristics. The conditions which must be satisfied on this surface by the quantities characterizing the physico-mechanical state of the system will be called the conditions of physico-mechanical contact of the bodies under consideration.

We should note that this formulation of the problem corresponds, in particular, to the connection of bodies by means of artificial layers. For example, this pertains to systems of bodies which are connected by means of welded seams, stiffening ribs, etc.

Let us point out the derivation of the conditions of physico-mechanical contact for the case when only mechanical and thermal processes occur in a system of elastic bodies. For purposes of simplification, we shall disregard the forces of inertia, and also the effect of thermoelastic dissipation -- i.e., we shall investigate the conditions of physico-mechanical contact for a quasistatic temperature problem of elasticity theory.

With respect to the conditions of mechanical contact, as was already noted, we may employ the results derived from the theory of shells, different variations of which have been adequately developed. We shall deal with the problem of thermal contact, which has been studied to a lesser extent. /212

Let us investigate a solid body containing a foreign inclusion. In accordance with the computational model, we shall assume that the body and the inclusion are divided by a thin intermediate layer, on the surfaces S_1 and S_2 of which there is ideal thermal contact with the inclusion and with the body, respectively.

Let k_1 and k_2 be the principal curvatures of the layer middle surface, referred to the curvature lines. Employing this surface as the base surface, we shall introduce an orthogonal mixed coordinate system (α, β, γ) into the region occupied by the layer. In order to determine the temperature t of the layer, in accordance with the general thermal conductivity theory (Ref. 8), we obtain:

The equation of thermal balance

$$\frac{\partial H_\beta J_\alpha}{\partial \alpha}+\frac{\partial H_\alpha J_\beta}{\partial \beta}+\frac{\partial}{\partial \gamma}(H_\alpha H_\beta J_\gamma)=-cH_\alpha H_\beta\frac{\partial t}{\partial \tau}, \tag{8}$$

where $H_\alpha = A(1 + k_1\gamma)$; $H_\beta = B(1 + k_2\gamma)$ are the Lamé coefficients; A, B -- coefficients of the first quadratic form of the surface S_0; c -- specific heat per unit volume; τ -- time);

Fourier equations

$$J_\alpha=\lambda X_\alpha;\quad J_\beta=\lambda X_\beta;\quad J_\gamma=\lambda X_\gamma, \tag{9}$$

relating the components J_α, J_β, J_γ of the thermal flux vector to the components of the thermodynamic force vector

$$X_\alpha=-\frac{1}{H_\alpha}\cdot\frac{\partial t}{\partial \alpha};\quad X_\beta=-\frac{1}{H_\beta}\cdot\frac{\partial t}{\partial \beta};\quad X_\gamma=-\frac{\partial t}{\partial \gamma}; \tag{10}$$

The boundary conditions

$$\begin{aligned} t&=t_1;\quad \lambda\frac{\partial t}{\partial n_1}=\lambda_1\frac{\partial t_1}{\partial n_1}\text{ on }\quad S_1;\\ t&=t_2;\quad \lambda\frac{\partial t}{\partial n_2}=\lambda_2\frac{\partial t_2}{\partial n_2}\text{ on }\quad S_2, \end{aligned} \tag{11}$$

where t_1, t_2 -- temperature; λ_1, λ_2 -- thermoconductivity of the inclusion and the layer, respectively; $\bar{n}_1$ and $\bar{n}_2$ -- normals to the surfaces S_1 and S_2;

Initial conditions

$$t = t_0 \text{ for } \tau = 0. \tag{12}$$

Taking the fact into account that the layer thickness 2h is small as compared with other dimensions, and regarding it as a thin shell, we may reduce the three-dimensional problem (Ref. 8-12) of the thermoconductivity theory to a two-dimensional problem, based on the hypothesis of linear temperature distribution over the layer thickness

$$t = T + \gamma\theta. \tag{13}$$

For this purpose, let us write the formation of entropy Σ per unit volume /213 as follows, in accordance with (9) (Ref. 3):

$$2\sum = -\lambda (X_\alpha^2 + X_\beta^2 + X_\gamma^2). \tag{14}$$

and also its increase

$$d\sum = J_\alpha dX_\alpha + J_\beta dX_\beta + J_\gamma dX_\gamma \tag{15}$$

Let us examine the quantity

$$\sigma = \int_{-h}^{h} (1 + k_1\gamma)(1 + k_2\gamma) \sum d\gamma, \tag{16}$$

which represents the entropy in a shell computed per unit area of its middle surface. It is also apparent that we have the following in the linear formulation

$$d\sigma = \int_{-h}^{h} (1 + k_1\gamma)(1 + k_2\gamma)\, d\sum d\gamma. \tag{17}$$

In accordance with (13), we obtain the following according to the formulas (10)

$$X_\alpha = \frac{x_1 + \xi_1\gamma}{1 + k_1\gamma}; \quad X_\beta = \frac{x_2 + \xi_2\gamma}{1 + k_2\gamma}; \quad X_\gamma = \theta, \tag{18}$$

where

$$x_1 = -\frac{1}{A}\cdot\frac{\partial T}{\partial \alpha}; \quad x_2 = -\frac{1}{B}\cdot\frac{\partial T}{\partial \beta}; \quad \xi_1 = -\frac{1}{A}\cdot\frac{\partial \theta}{\partial \alpha}; \quad \xi_2 = -\frac{1}{B}\cdot\frac{\partial \theta}{\partial \beta}. \tag{19}$$

First substituting (18) into (14), (15), and the results obtained into (16), (17), respectively, and disregarding terms k_1h, k_2h as compared with unity after integration, we obtain

$$\sigma = -\left[\lambda h (x_1^2 + x_2^2 + \theta^2) + \frac{\lambda h^3}{3}(\xi_1^2 + \xi_2^2)\right]; \tag{20}$$

$$d\sigma = I_1 dx_1 + I_2 dx_2 + L_1 d\xi_1 + L_2 d\xi_2 + I_3 d\theta, \tag{21}$$

where

$$I_1 = \int_{-h}^{h} (1 + k_2\gamma) J_\alpha d\gamma; \quad L_1 = \int_{-h}^{h} (1 + k_2\gamma) J_\alpha d\gamma;$$
$$I_2 = \int_{-h}^{h} (1 + k_1\gamma) J_\beta d\gamma; \quad L_2 = \int_{-h}^{h} (1 + k_1\gamma) J_\beta d\gamma. \tag{22}$$

It directly follows from (20), (21), and (19) that

$$I_1 = -2\lambda h x_1 = -2\lambda h \frac{1}{A}\cdot\frac{\partial T}{\partial\alpha};\ L_1 = -\frac{2}{3}\lambda h^3 \xi_1 = -\frac{2}{3}\lambda h^3 \frac{1}{A}\cdot\frac{\partial\theta}{\partial\alpha};$$
$$I_2 = -2\lambda h x_2 = -2\lambda h \frac{1}{B}\cdot\frac{\partial T}{\partial\beta};\ L_2 = -\frac{2}{3}\lambda h^3 \xi_2 = -\frac{2}{3}\lambda h^3 \frac{1}{B}\cdot\frac{\partial\theta}{\partial\beta}. \qquad (23)$$

Let us now rewrite equation (8) in the following form /214

$$\frac{\partial H_\beta J_\alpha}{\partial\alpha} + \frac{\partial H_\alpha J_\beta}{\partial\beta} - \lambda\frac{\partial}{\partial\gamma}\left(H_\alpha H_\beta\frac{\partial t}{\partial\gamma}\right) = -cH_\alpha H_\beta\frac{\partial t}{\partial\tau} \qquad (24)$$

and let us integrate it over γ from $-h$ to h.

Let us multiply this equation by γ, and let us again perform integration within the same limits. As a result, we obtain the following, disregarding terms of the order k_1h_1, k_2h as compared with unity after integration, and considering (23)

$$\lambda_0\Delta T + \lambda\left[\left(\frac{\partial t}{\partial\gamma}\right)^+ - \left(\frac{\partial t}{\partial\gamma}\right)^-\right] = -c_0\left(\frac{\partial T}{\partial\tau} + \frac{2}{3}Hh^2\frac{\partial\theta}{\partial\tau}\right);$$
$$\lambda_0 h\Delta\theta + 3\lambda\left[\left(\frac{\partial t}{\partial\gamma}\right)^+ + \left(\frac{\partial t}{\partial\gamma}\right)^-\right] - \frac{6}{r_0}\left(t^+ - t^- - 4HhT - \right.$$
$$\left. -\frac{4}{3}Kh^3\theta\right) = c_0\left(h\frac{\partial\theta}{\partial\tau} + \frac{1}{6}Hh\frac{\partial T}{\partial\tau}\right), \qquad (25)$$

where $\lambda_0 2\lambda h$ is thermoconductivity; $c_0 = 2ch$ -- specific heat of the layer; $r_0 = \frac{2h}{\lambda}$ -- its thermal resistance; $H = \frac{1}{2}(k_1 + k_2)$ -- average curvature, and $K = k_1k_2$ -- Gaussian curvature of the layer middle surface;

$$\Delta = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\cdot\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\cdot\frac{\partial}{\partial\beta}\right)\right] \qquad (26)$$

(the plus and minus indices indicate that the quantities are taken in the case $\gamma = \pm h$, correspondingly).

Integrating the initial condition (12) in a similar way, we obtain

$$T = T_0;\quad h\theta = \theta_0 \text{ for } \tau = 0, \qquad (27)$$

where

$$T_0 = \frac{1}{2h}\int_{-h}^{h} t_0\, d\gamma;\quad \theta_0 = \frac{3}{2h^2}\int_{-h}^{h} t_0\gamma\, d\gamma. \qquad (28)$$

In addition, we obtain the following from (13)

$$T = \frac{1}{2}(t^+ + t^-);\quad \theta = \frac{1}{2h}(t^+ - t^-). \qquad (29)$$

Let us now omit T and θ from (25), (27), (29), taking the boundary conditions (11) into account, and in the relationships obtained we find that in the case $h \to 0$, λ_0, r_0, c_0 are constants.

As a result, we obtain the desired conditions of thermal contact in the following form /215

$$\left.\begin{aligned} &\lambda_0\Delta(t_1 + t_2) + 2\left(\lambda_1\frac{\partial t_1}{\partial n_1} + \lambda_2\frac{\partial t_2}{\partial n_2}\right) = c_0\frac{\partial(t_1+t_2)}{\partial\tau};\\ &\lambda_0\Delta(t_1 - t_2) + 6\left(\lambda_1\frac{\partial t_1}{\partial n_1} - \lambda_2\frac{\partial t_2}{\partial n_2}\right) - \frac{12}{r_0}(t_1 - t_2) = \\ &= c_0\frac{\partial(t_1 - t_2)}{\partial\tau};\end{aligned}\right\}\text{on } S_0; \qquad (30)$$

$$t_1 + t_2 = 2T_0;\quad t_1 - t_2 = t_0 \text{ for } \tau = 0. \qquad (31)$$

Conditions (30), (31) were obtained (Ref. 16, 17) by a different method, and they were employed in (Ref. 5) to solve the plane problem of thermoelasticity for an infinite plate with an elastic circular disk. The influence of the intermediate layer on the disturbance of the uniform thermal flux in the vicinity of the spherical inclusion in an infinite body was studied in (Ref. 20), in which the results obtianed in (Ref. 24) were generalized for a spherical cavity with an insulated surface.

Equation (30) leads to different particular cases of the formulation of boundary conditions for the thermoconductivity problem. For example, assuming $\lambda_1 = \lambda_2 = \lambda$, $\bar{n}_1 = -\bar{n}_2 = \bar{n}$, we obtain the conditions

$$\begin{gathered}\lambda_0 \Delta (t_1 + t_2) + 2\lambda \frac{\partial (t_1 - t_2)}{\partial n} = c_0 \frac{\partial (t_1 + t_2)}{\partial \tau};\\ \lambda_0 \Delta (t_1 - t_2) + 6\lambda \frac{\partial (t_1 + t_2)}{\partial n} - \frac{12}{r_0}(t_1 - t_2) = c_0 \frac{\partial (t_1 - t_2)}{\partial \tau},\end{gathered} \quad (32)$$

which correspond to surface inclusion in a uniform body.

If we assume that the contact thermoconductivity $\lambda_0 = 0$, instead of (30), we shall have

$$\begin{gathered}\lambda_1 \frac{\partial t_1}{\partial n_1} + \lambda_2 \frac{\partial t_2}{\partial n_2} = \frac{c_0}{2} \frac{\partial (t_1 + t_2)}{\partial \tau};\\ \lambda_1 \frac{\partial t_1}{\partial n_1} - \lambda_2 \frac{\partial t_2}{\partial n_2} - \frac{2}{r_0}(t_1 - t_2) = c_0 \frac{\partial (t_1 - t_2)}{\partial \tau}.\end{gathered} \quad (33)$$

Also assuming the contact specific heat $c_0 = 0$, we obtain

$$\lambda_1 \frac{\partial t_1}{\partial n_1} + \lambda_2 \frac{\partial t_2}{\partial n_2} = 0; \quad \lambda_1 \frac{\partial t_1}{\partial n_1} - \lambda_2 \frac{\partial t_2}{\partial n_2} = \frac{2}{r_0}(t_1 - t_2). \quad (34)$$

Combining and subtracting the last two relationships we obtain the conditions

$$\frac{\partial t_1}{\partial n_1} - \frac{1}{\lambda_1 r_0}(t_1 - t_2) = \frac{\partial t_2}{\partial n_2} + \frac{1}{\lambda_2 r_0}(t_1 - t_2) = 0,$$

The form of these conditions coincides with that of the conditions given in (Ref. 26, 27).

Omitting $\lambda_2 \frac{\partial t_2}{\partial n_2}$ from equations (30) and assuming $t_1 = t$, $t_2 = t_c$, we obtain the Newton condition /216

$$\lambda \frac{\partial t}{\partial n} + \alpha (t - t_c) = 0,$$

where $\alpha = \frac{1}{r_0}$ is the absolute heat transfer coefficient determined as the inverse value of the thermal resistance of the boundary layer.

Finally, in the case of the contact resistance $r_0 = 0$, we arrive at the conditions of ideal thermal contact

$$\lambda_1 \frac{\partial t_1}{\partial n_1} + \lambda_2 \frac{\partial t_2}{\partial n_2} = 0; \quad t_1 = t_2.$$

In conclusion, we would like to note that these conditions may be directly employed in the case of a plane temperature field, if it is assumed that t_1 and

t_2 are functions of only two three-dimensional coordinates, and if we set $\Delta = \frac{\partial^2}{\partial s^2}$, where s is the arc length of the conjugate profile L_0. For a generalized plane field, the corresponding conditions will be different due to heat exchange on the end faces of the intermediate layer. If t_1 and t_2 are the averaged plate temperatures, then the following relationships must be fulfilled on the profile L_0 dividing the regions of plates made of different materials which are combined at the joint by means of a thin intermediate layer having the thickness 2δ:

$$\lambda_0 \frac{\partial^2 (t_1+t_2)}{\partial s^2} + 2\left(\lambda_1 \frac{\partial t_1}{\partial n} - \lambda_2 \frac{\partial t_2}{\partial n}\right) - \frac{\alpha_0}{2}(t_1+t_2) = c_0 \frac{\partial (t_1+t_2)}{\partial \tau} - \alpha T_c;$$
$$\lambda_0 \frac{\partial^2 (t_1-t_2)}{\partial s^2} + 6\left(\lambda_1 \frac{\partial t_1}{\partial n} + \lambda_2 \frac{\partial t_2}{\partial n}\right) - \left(\frac{\alpha_0}{2} + \frac{6}{r_0}\right)(t_1-t_2) = c_0 \frac{\partial (t_1-t_2)}{\partial \tau} - \alpha T_c^*, \tag{35}$$

where $\lambda_0 = \lambda S$; $c_0 = cS$; $S = 4h\delta$ is the area of transverse cross-section of the intermediate layer; $\alpha_0 = \alpha \ell_c$; $\ell_c = 4\delta$ -- the external, circumfluous middle portion of the profile for the layer transverse cross-section $r_0 = \frac{1}{\lambda} \cdot \frac{\ell_c}{\ell_k}$; ℓ_k -- the portion of the profile of the layer transverse cross-section in which it makes contact with the joined plates; $\lambda_1^0 = 2\lambda_1 h$; $\lambda_2^0 = 2\lambda_2 h$.

It is apparent that, with this formulation of the thermophysical contact parameters, the conditions obtained may also be employed to study the temperature fields in plates with stiffening ribs. As a particular case, relationships (35) follow from the corresponding general equations for plates and shells which were obtained in (Ref. 2). /217

If the plate edge is reinforced by a thin rod made of another material having the same thickness 2h and the width 2δ, in order to determine the generalized plane field, we shall have (Ref. 4) the following conditions of heat exchange on the reinforced edge L_0

$$\left[\lambda_0 \frac{\partial^2}{\partial s^2} + \left(1 + \frac{1}{2}\alpha_c^0 r_0\right)\lambda_n^0 \frac{\partial}{\partial n} - c_0 \frac{\partial}{\partial \tau}\right] t = (\alpha_c^0 + \alpha_0)(t - t_c), \tag{36}$$

which are characterized by the four thermophysical rod parameters: thermal resistance r_0, thermal conductivity λ_0, specific heat c_0, heat transfer α_0. Here we have $\alpha_c^0 = d_c \ell_c$; $\lambda_n^0 = \lambda_n^0 = \lambda_n \ell_k$.

Setting $\alpha_c = \alpha_0$ in (36), we obtain the condition

$$\left[\lambda_0 \frac{\partial^2}{\partial s^2} + \left(1 + \frac{1}{2}\alpha_c^0 r_0\right)\lambda_n^0 \frac{\partial}{\partial n} - c_0 \frac{\partial}{\partial \tau}\right] t = \alpha_c l (t - t_c), \tag{37}$$

where $\ell = \ell_c + \ell_k$.

This condition may also be applied for plates whose edge is reinforced by a thin rod having an arbitrary cross-section, and ℓ_c then designates the outer, circumfluous middle portion of the profile for the rod transverse cross-section.

It may be readily seen that the well known condition of the Newton heat exchange for an unreinforced plate follows from (36), (7) for zero values of all four thermophysical rod parameters.

Setting $r_c = \frac{1}{\alpha_c} = 0$ in (36), we obtain the following condition

$$\lambda \frac{\partial t}{\partial n} = \frac{t - t_c}{r_k^*}, \tag{38}$$

which coincides with the Newton condition for an unreinforced plate. In this condition, the thermoresistance of the rod r_k^* plays the role of the heat exchange resistance r_c on the plate surface. We obtain the well known condition of the first kind form (38) in the case $r_k^* = 0$.

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3 EFFECT OF A DIFFUSION PROCESS ON THE STRESS CONCENTRATION NEAR A CIRCULAR HOLE /219

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The well known book by G. N. Savin[1] made a detailed investigation of stress concentration around holes as viewed from the classical elasticity theory. The study of stress concentration in a body representing a solid solution when the activity of the diffusion process is quite great is of significant importance. It may be assumed that the substance is redistributed due to the influence of large stress gradients, and this redistribution may lead to a change in the stress state of the body. We obtained a system of differential equations reflecting the interrelationship between the processes of deformation, diffusion, and thermoconductivity. This sytem of equations enabled us to study the influence of diffusion produced by a nonuniform stress state upon the change in the time of stress concentration around holes.

This problem may be reduced to solving the following equations at a constant temperature

$$D\Delta c = \frac{\partial c}{\partial \tau}; \tag{1}$$

$$\Delta\Delta U = 0; \tag{2}$$

$$\Delta\psi = E\beta_c c \tag{3}$$

in the case of a boundary condition for the concentration in the following form

$$D_c \operatorname{grad}_n c + D_\sigma \operatorname{grad}_n \sigma' = -H(\mu - \mu_c) \tag{4}$$

and for stresses which are specifically defined on the boundary surface. Here $D = D_c - E\beta_c D\sigma$; H -- mass exchange coefficient.

Let us investigate an infinite body which represents a solid solution /220
having a constant concentration c_0 with a circular, cylindrical cavity having the radius R in the case of unidirectional tension at infinity by forces p.

Introducing the polar coordinates (r, θ), the dimensionless time $\tau_1 = \tau \frac{D}{R^2}$, and applying the Laplace transformation to relationships (1), (4), we obtain

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\cdot\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\cdot\frac{\partial^2}{\partial\theta^2}\right)\tilde{c}(\rho, \theta, s) - s\tilde{c}(\rho, \theta, s) = 0; \tag{5}$$

$$\tilde{c}(\rho, \theta, \infty) = 0; \quad c(\infty, \theta, s) = 0; \quad \left.\frac{\partial \tilde{c}(\rho, \theta, s)}{\partial\rho}\right|_{\rho\to\infty} = 0;$$

$$D_c \left.\frac{\partial \tilde{c}(\rho, \theta, s)}{\partial\rho}\right|_{\rho=1} + D_\sigma \left.\frac{\partial \tilde{\sigma}(\rho, \theta, s)}{\partial\rho}\right|_{\rho=1} = 0, \tag{6}$$

where $\tilde{c}$, $\tilde{\sigma}$ are the Laplace transforms respectively of the quantities $c_1 = c - c_0$, $\sigma' = \sigma_{rr} + \sigma_{\theta\theta}$. The boundary condition (6) corresponds to the absence

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[1] Savin, G.N. Kontsentratsiya napryazheniy okolo otverstiy (Stress Concentration Around Holes). Moscow-Leningrad, Gostekhixdat, 1951.

of a diffusion flux of solute in the cavity.

The equations for determining the Laplace transform $\tilde{U}$, $\tilde{\psi}$ of the functions U, ψ may be written as follows

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\cdot\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\cdot\frac{\partial^2}{\partial\theta^2}\right)\tilde{U}(\rho,\theta,s)=0; \quad (7)$$

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\cdot\frac{\partial^2}{\partial\theta^2}\right)\psi(\rho,\theta,s)=E\beta_c\tilde{c}, \quad (8)$$

and the stress representations may be determined by the following relationships according to formulas (3.3) given in (Ref. 2):

$$\tilde{\sigma}_{rr}=\left(\frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\cdot\frac{\partial^2}{\partial\theta^2}\right)(\tilde{U}-\tilde{\psi}); \quad \tilde{\sigma}_{\theta\theta}=\frac{\partial^2}{\partial\rho^2}(\tilde{U}-\tilde{\psi});$$
$$\sigma_{r\theta}=\left(\frac{1}{\rho^2}\cdot\frac{\partial}{\partial\theta}-\frac{1}{\rho}\cdot\frac{\partial^2}{\partial\rho\partial\theta}\right)(\tilde{U}-\tilde{\psi}). \quad (9)$$

The boundary conditions for the stresses are as follows:

In the case $\rho \to \infty$

$$\tilde{\sigma}_{rr}=\frac{1}{2}\cdot\frac{p}{s}(1+\cos 2\theta); \quad \tilde{\sigma}_{r\theta}=-\frac{1}{2}\cdot\frac{p}{s}\sin 2\theta; \quad (10)$$

In the case $\rho = 1$

$$\tilde{\sigma}_{rr}=0; \quad \sigma_{r\theta}=0. \quad (11)$$

A solution of equation (5) satisfying conditions (6) has the following form

$$\tilde{c}(\rho,\theta,s)=AK_2(\rho\sqrt{s})\cos 2\theta, \quad (12)$$

where $K_2(\rho\sqrt{s})$ is the Bessel function.

We may select the functions U and $\tilde{\psi}$ which satisfy equations (7) and (8) in the following form /221

$$\tilde{U}=a_0\rho^2+b_0\ln\rho+(a_1\rho^2+a_2+a_3\rho^{-2})\cos 2\theta;$$
$$\tilde{\psi}=\frac{1}{s}E\beta_c\tilde{c}. \quad (13)$$

Substituting the values of the functions $\tilde{U}$ and $\tilde{\psi}$ from (13) in relationship (9), we obtain

$$\tilde{\sigma}_{rr}=2a_0+\frac{b_0}{\rho^2}-(2a_1+4a_2\rho^{-2}+6a_3\rho^{-4})\cos 2\theta+$$
$$+\gamma A\left[\frac{6}{\rho^2 s}K_2(\rho\sqrt{s})+\frac{1}{\rho\sqrt{s}}K_1(\rho\sqrt{s})\right]\cos 2\theta; \quad (14a)$$

$$\tilde{\sigma}_{\theta\theta}=2a_0-\frac{b_0}{\rho^2}+(2a_1+6a_3\rho^{-4})\cos 2\theta-\gamma A\left[\frac{6}{\rho^2 s}K_2(\rho\sqrt{s})+\right.$$
$$\left.+\frac{3}{\rho\sqrt{s}}K_1(\rho\sqrt{s})+K_0(\rho\sqrt{s})\right]\cos 2\theta; \quad (14b)$$

$$\tilde{\sigma}_{r\theta}=(2a_1-2a_2\rho^{-2}-6a_3\rho^{-4})\sin 2\theta+ \quad (14c)$$
$$+\gamma A\left[\frac{6}{\rho^2 s}K_2(\rho\sqrt{s})+\frac{2}{\rho\sqrt{s}}K_1(\rho\sqrt{s})\right]\sin 2\theta; \quad \gamma=E\beta_c.$$

The unknown constants a_0, b_0, a_1, a_2, a_3 may be determined in terms of p and A from the boundary conditions (10) and (11). Determining the following quantity by means of the first two relationships (14)

$$\tilde{\sigma}=\frac{p}{s}+\left[\frac{2\gamma A}{\sqrt{s}}K_1(\sqrt{s})-\frac{2p}{s}\right]\frac{\cos 2\theta}{\rho^2}-\gamma AK_2(\rho\sqrt{s})\cos 2\theta$$

and substituting its value in the boundary condition (6), we find the desired quantity

$$A=\frac{b_1}{s\left[\left(\frac{b_2}{\sqrt{s}}+\sqrt{s}\right)K_1(\sqrt{s})+2K_0(\sqrt{s})\right]}, \tag{15}$$

where

$$b_1=\frac{4pD_a}{D_c-E\beta_c D_a};\quad b_2=\frac{4D_c}{D_c-E\beta_c D_a}. \tag{16}$$

With allowance for (15), the inverse Laplace transformation of expression (11) yields

$$c(\rho,\theta,\tau_1)=c_0+\frac{b_1}{b_2}\cdot\frac{\cos 2\theta}{\rho^2}-2b_1 I_1(\rho,\tau_1)\cos 2\theta, \tag{17}$$

where

$$I_1(\rho,\tau_1)=\frac{1}{\pi}\int_0^\infty\frac{[Y_2(\rho t)f_1(t)-J_2(\rho t)f_2(t)]e^{-t^2\tau_1}dt}{f_1^2(t)+f_2^2(t)};$$

$$f_1(t)=(b_2-t^2)J_1(t)-2tJ_0(t),\quad f_2(t)=(b_2-t^2)Y_1(t)-2tY_0(t),$$

and J, Y are Bessel functions.

Similarly, we may obtain the stresses influencing the body /222

$$\sigma_{rr}=\frac{p}{2}\left(1-\frac{1}{\rho^2}\right)+\frac{p}{2}\left(1-\frac{4}{\rho^2}+3\rho^{-4}\right)\cos 2\theta+2\gamma b_1[I_3(\rho,\tau_1)+$$
$$+2I_2(\rho,\tau_1)]\cos 2\theta-2\gamma b_1\rho^{-2}\left[2I_2(1,\tau_1)-\frac{3}{\rho^2}I_2(1,\tau_1)+\right. \tag{18a}$$
$$\left.+\frac{1}{\rho^2}I_3(1,\tau_1)\right]\cos 2\theta;$$

$$\sigma_{\theta\theta}=\frac{p}{2}\left(1+\frac{1}{\rho^2}\right)-\frac{p}{2}(1+3\rho^{-4})\cos 2\theta-2\gamma b_1\rho^{-4}[3I_2(1,\tau_1)- \tag{18b}$$
$$-I_3(1,\tau_1)]\cos 2\theta-2\gamma b_1[I_3(\rho,\tau_1)-I_2(\rho,\tau_1)-\rho I_1(\rho,\tau_1)]\cos 2\theta;$$

$$\sigma_{r\theta}=-\frac{p}{2}(1+2\rho^{-2}-3\rho^{-4})\sin 2\theta+2\gamma b_1[I_3(\rho,\tau_1)-$$
$$-2I_2(\rho,\tau_1)]\sin 2\theta-2\gamma b_1\rho^{-2}\left[I_2(1,\tau_1)-\frac{3}{\rho^2}I_2(1,\tau_1)+\right. \tag{18c}$$
$$\left.+\frac{1}{\rho^2}I_3(1,\tau_1)\right]\sin 2\theta,$$

where

$$I_2(\rho,\tau_1)=\frac{1}{\pi}\int_0^\infty\frac{[Y_1(\rho t)f_1(t)-J_1(\rho t)f_2(t)]e^{-t^2\tau_1}dt}{\rho t[f_1^2(t)+f_2^2(t)]};$$

$$I_3(\rho,\tau_1)=\frac{6}{\pi}\int_0^\infty\frac{[Y_2(\rho t)f_1(t)-J_2(\rho t)f_2(t)]e^{-t^2\tau_1}dt}{\rho^2t^2[f_1^2(t)+f_2^2(t)]}.$$

For a stationary regime ($\tau_1\to\infty$), we have the following from formulas (17) and (18a), (18b), (18c):

$$c(\rho,\theta)=c_0+\frac{b_1}{b_2}\frac{\cos 2\theta}{\rho^2},$$

and the stresses do not depend on the solute concentration. For small values of τ_1, the formulas obtained may be written approximately as follows:

$$c(\rho,\theta,\tau_1)=c_0+\frac{b_1\cos 2\theta}{\sqrt{\rho}}\left[B_1(\rho,\tau_1)+\frac{15}{8}\sqrt{\frac{1}{\rho}}B_2(\rho,\tau_1)\right]; \tag{19}$$

$$\sigma_{rr}=\frac{p}{2}(1-\rho^{-2})+\frac{p}{2}(1-4\rho^{-2}+3\rho^{-4})\cos 2\theta+$$

$$+\gamma b_1\rho^{-1}\left[\frac{2}{\rho}\left(1-\frac{3}{2\rho^2}\right)B_2(1,\tau_1)+\sqrt{\frac{1}{\rho}}\,B_2(\rho,\tau_1)\right]\cos 2\theta; \tag{20a}$$

$$\sigma_{\theta\theta}=\frac{p}{2}(1+\rho^{-2})-\frac{p}{2}(1+3\rho^{-4})\cos 2\theta+\gamma b_1\left[\frac{3}{\rho^4}B_2(1,\tau_1)-\right.$$
$$\left.-\sqrt{\frac{1}{\rho}}\,B_1(\rho,\tau_1)-\frac{23}{8}\rho^{-3/2}B_2(\rho,\tau_1)\right]\cos 2\theta; \tag{20b}$$

$$\sigma_{r\theta}=-\frac{p}{2}(1+2\rho^{-2}-3\rho^{-4})\sin 2\theta+$$
$$+\gamma b_1\frac{\sin 2\theta}{\rho}\left[\frac{1}{\rho}\left(1-\frac{3}{\rho^2}\right)B_2(1,\tau_1)+\frac{2}{\sqrt{\rho}}\,B_2(\rho,\tau_1)\right], \tag{20c}$$

where /223

$$B_1(\rho,\tau_1)=\frac{8}{19}\left\{\operatorname{erfc}\frac{\rho-1}{2\sqrt{\tau_1}}-\exp\left[\frac{19}{8}(\rho-1)+\right.\right.$$
$$\left.\left.+\frac{19^2}{8^2}\tau_1\right]\operatorname{erfc}\left[\frac{19}{8}\sqrt{\tau_1}+\frac{\rho-1}{2\sqrt{\tau_1}}\right]\right\};$$

$$B_2(\rho,\tau_1)=\frac{8}{19}\left\{2\sqrt{\frac{\tau_1}{\pi}}\exp\left[-\frac{(\rho-1)^2}{4\tau_1}\right]-\left(\frac{8}{19}+\rho-1\right)\operatorname{erfc}\frac{\rho-1}{2\sqrt{\tau_1}}+\right.$$
$$\left.+\frac{8}{19}\exp\left[\frac{19}{8}(\rho-1)+\frac{19^2}{8^2}\tau_1\right]\operatorname{erfc}\left[\frac{\rho-1}{2\sqrt{\tau_1}}+\frac{19}{8}\sqrt{\tau_1}\right]\right\}.$$

Let us study the change with time of the solute concentration and the stresses at the point $\rho = 1$, $\theta = \frac{\pi}{2}$ and at the point which is symmetrical to it, in which we obtain the maximum stress concentration from the solution of the Kirsch problem. We have the following from formulas (17), (18a), (18b), (18c)

$$c\left(1,\frac{\pi}{2},\tau_1\right)=c_0-\frac{b_1}{b_2}-\frac{4b_1}{\pi^2}\int_0^\infty\frac{(b_2-t^2-4)\,e^{-t^2\tau_1}\,dt}{t\,[f_1^2(t)+f_2^2(t)]}; \tag{21}$$

$$\sigma_{\theta\theta}\left(1,\frac{\pi}{2},\tau_1\right)=\sigma_{\theta\theta}^0+\sigma_{\theta\theta}^1=3p+\frac{4\gamma b_1}{\pi^2}\int_0^\infty\frac{(b_2-t^2)\,e^{-t^2\tau_1}\,dt}{t\,[f_1^2(t)+f_2^2(t)]}. \tag{22}$$

However, for this same case we may obtain the following approximate expressions from (19), (20a), (20b), (20c)

$$c\left(1,\frac{\pi}{2},\tau_1\right)=c_0-\frac{8}{19}b_1\left\{\frac{4}{19}\left[1-\exp\left(\frac{19^2}{8^2}\tau_1\right)\operatorname{erfc}\left(\frac{19}{8}\sqrt{\tau_1}\right)\right]+\frac{15}{4}\sqrt{\frac{\tau_1}{\pi}}\right\}; \tag{23}$$

(23)

$$\sigma_{\theta\theta}\left(1,\frac{\pi}{2},\tau_1\right)=3p+\frac{8\gamma b_1}{19}\left\{\frac{20}{19}\left[1-\exp\left(\frac{19^2}{8^2}\tau_1\right)\operatorname{erfc}\left(\frac{19}{8}\sqrt{\tau_1}\right)\right]-\frac{2}{8}\sqrt{\frac{\tau_1}{\pi}}\right\}. \tag{24}$$

All the formulas presented above are suitable for the case of plane deformation, if we assume

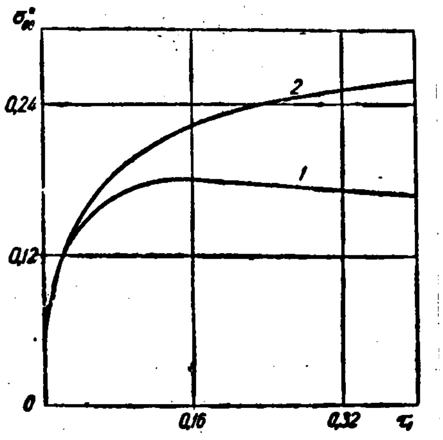
$$b_1^*=\frac{4pD_\sigma^*}{D_c^*-E^*\beta_c^*D_\sigma^*};\quad b_2^*=\frac{4D_c^*}{D_c^*-E^*\beta_c^*D_\sigma^*};\quad \gamma^*=E^*\beta_c^*. \tag{25}$$

instead of b_1, b_2, γ.

We should point out that, since the concentration coefficient of linear expansion $\beta_c(\beta_c^*)$ may be positive or negative depending on the type of solid solution and $D_\sigma = -\frac{1}{\rho}\beta_c L$, $L>0$, $D_c - E\beta_c D_\sigma>0$, the constant $b_1(b_1^*)$ has the minus or plus sign, respectively, and the product $\gamma b_1(\gamma^* b^*)$ is always negative.

As may be readily seen from formulas (16) and (25), the constant $b_2(b_2^*)$ changes /224 from 0 to 4.

As follows from the approximate formulas (23) and (24), depending upon whether β_c is smaller or larger than zero, in the most extended zones the solute concentration increases or decreases with time, and the effective stresses in both cases decrease.



The figure presents a graph showing the change in the concentration stress $\sigma_{\theta\theta}^* = -\frac{\sigma_{\theta\theta}^{(1)}(\tau_1)}{\gamma^{b_1}}$ with time; this was calculated from a precise formula (22) (curve 1) and from an approximate formula (24) (curve 2) in the case $b_2 = 3$. The change in the quantity $\sigma_{\theta\theta}^*$ computed according to the precise formula (curve 1) shows that the concentration stresses first increase, reach the largest value, then slowly decrease, and strive to zero in the case $\tau \to \infty$.

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PROPAGATION OF ELASTIC WAVES ALONG A CYLINDRICAL CAVITY FILLED WITH A CONDUCTING FLUID

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This article investiagtes the magnetohydroelasticity problem of axisymmetric wavesin an elastic body-conductive liquid system. A cylindrical cavity having a circular transverse cross-section, filled with a nonviscous, compressible conductive liquid, exists in an elastic body having infinite dimensions. A uniform constant magnetic field is applied along the cavity axis. It is assumed that the elastic body is magnetically and electrically neutral. In essence, this system represents a wave guide having an unusual characteristic. This type of problem is important for geophysical research.

Several articles (Ref. 7-11) have investigated steady magnetohydrodynamic motion of liquid in channels with rigid walls, or without rigid walls. Several of these articles have advanced definite assumptions regarding either the strength component of the perturbed magnetic field or regarding the compressibility of the conductive medium. This makes it possible to obtain the solution containing Bessel functions. However, a more precise formulation of the problem leads to an investigation of degenerate hypergeometric equations (Ref. 7).

The notation is as following: $\vec{H}$ -- vector of the magnetic field strength; $\vec{h}$ -- vector of the perturbed magnetic field strength; $\vec{E}$ -- vector of the electric field strength; $\vec{j}$ -- current density; $\vec{v}$ -- velocity vector of a liquid particle; ρ -- liquid density; p -- liquid pressure; μ -- magnetic permeability; σ -- electroconductivity; c_b -- speed of light; c_0 -- speed of sound in a nonconductive liquid; t -- time; x -- axial coordinate; r -- radial coordinate; /226 r_0 -- cavity radius; ϕ -- elastic scalar potential; ψ -- modulus of elastic vector potential; σ_r -- radial stress; τ -- shearing stress; R -- radial displacement; γ -- elastic medium density; ν -- Poisson coefficient; c_s -- velocity of distortion wave in an elastic medium; c_e -- expansion wave velocity in an elastic medium; $a^2 = \frac{\mu H^2}{4\pi\rho_0}$ -- square of the Alfvén wave velocity; c -- phase velocity; ℓ -- wavelength.

The dimensionless quantities are as follows:

$$\left.\begin{aligned}&(x^*,\ r^*)=\frac{1}{r_0}(x,\ r);\quad t^*=\frac{c_s}{r_0}t;\ (p^*,\ \sigma_r^*)=\frac{1}{\gamma c_s^2}(p,\ \sigma_r);\ R^*=\frac{R}{r_0};\\&v^*=\frac{v}{c_0};\ (H^*,\ h^*)=\frac{1}{H_0}(H,\ h);\ c^*=\frac{c}{c_s};\ l^*=\frac{l}{r_0};\ s=\frac{2\pi}{l^*};\\&\omega=2\pi\frac{c^*}{l^*};\ \overline{m}=\frac{a^2}{c_s^2}.\end{aligned}\right\}\qquad\text{(A)}$$

Disregarding displacement currents, mass forces, and viscosity, and assuming that μ = 1 and σ = const., we may write the equations of magnetohydrodynamics (Ref. 5):

$$\operatorname{rot}\vec{H}=\frac{4\pi}{c_b}\vec{j};\ \operatorname{div}\vec{j}=0;\qquad(1)$$

$$\text{rot}\,\vec{E} = -\frac{1}{c_b}\cdot\frac{\partial\vec{H}}{\partial t};\ \text{div}\vec{H} = 0; \tag{2}$$

$$\vec{j} = \sigma\left(\vec{E} + \frac{1}{c_b}[\vec{v}\times\vec{H}]\right); \tag{3}$$

$$\rho\frac{d\vec{v}}{dt} = -\text{grad}\,p + \frac{1}{c_b}[\vec{j}\times\vec{H}]; \tag{4}$$

$$\frac{\partial\rho}{\partial t} + \text{div}\,(\rho\vec{v}) = 0; \tag{5}$$

$$\frac{\partial\rho}{\partial t} = \frac{1}{c_e^2}\cdot\frac{\partial p}{\partial t}. \tag{6}$$

The system of equations (1) - (6) may be reduced to two vector equations for $\vec{H}$ and $\vec{v}$, if the nonlinear hydrodynamic terms are disregarded:

$$\frac{\partial\vec{H}}{\partial t} = \text{rot}\,[\vec{v}\times\vec{H}] - \frac{c_b^2}{4\pi\sigma}\,\text{rot rot}\,\vec{H}; \tag{7}$$

$$\text{grad div}\,\vec{v}\frac{1}{c_e^2}\cdot\frac{\partial^2\vec{v}}{\partial t^2} - \frac{1}{4\pi\rho_0 c_e^2}\cdot\frac{\partial}{\partial t}[\text{rot}\,\vec{H}\times\vec{H}]. \tag{8}$$

The following assumptions will be advanced below: /227

1. A constant magnetic field having the strength $\vec{H}_0$ is in operation along the cavity axis.

2. The perturbed field $\vec{h}(x, r, t)$ is small as compared with $\vec{H}_0$:

$$|\vec{h}_{\max}| \ll |\vec{H}_0|;\quad \vec{H} = \vec{H}_0 + \vec{h}.$$

3. The motion is axisymmetrical.

4. The cavity walls are nonconductive. Therefore, the current density component, which is normal to the profile, equals zero: $j_r\big|_{r = r_0} = 0$.

If we disregard small nonlinear terms and introduce the dimensionless quantities according to formulas (A), under the assumptions advanced above, equations (7) and (8) may be reduced to four scalar equations for h_x^*, h_r^*, v_r^* and v_r^* :

$$\frac{\partial h_r^*}{\partial t^*} = H^* c_0^* \frac{\partial v_r^*}{\partial x^*} + \frac{c_b^2}{4\pi\sigma r_0 c_s}\cdot\frac{\partial^2 h_r^*}{\partial x^*} - \frac{c_b^2}{4\pi\sigma r_0 c_s}\cdot\frac{\partial^2 h_x^*}{\partial x^*\partial r^*}; \tag{9}$$

$$\frac{\partial h_x^*}{\partial t^*} = -H^* c_0^* \frac{1}{r^*}\cdot\frac{\partial}{\partial r^*}(r^* v_r^*) - \frac{c_b^2}{4\pi\sigma r_0 c_s}\cdot\frac{\partial}{\partial x^*}\cdot\frac{1}{r^*}\cdot\frac{\partial}{\partial r^*}(r^* h_r^*) + \frac{c_b^2}{4\pi\sigma r_0 c_s}\cdot\frac{1}{r^*}\cdot\frac{\partial}{\partial r^*}r^*\frac{\partial h_x^*}{\partial t^*}; \tag{10}$$

$$\frac{\partial}{\partial r^*}\cdot\frac{1}{r^*}\cdot\frac{\partial}{\partial r^*}(r^* v_r^*) + \frac{\partial^2 v_x^*}{\partial x^*\partial r^*} = \left(\frac{c_s}{c_0}\right)^2\frac{\partial^2 v_r^*}{\partial t^{*2}} - \frac{H_0^2 c_s}{4\pi\rho_0 c_e^2}H^*\frac{\partial^2 h_r^*}{\partial t^*\partial x^*} + \frac{H_0^2 c_s}{4\pi\rho_0 c_e^2}H^*\frac{\partial^2 h_x^*}{\partial t^*\partial r^*}; \tag{11}$$

$$\frac{1}{r^*}\cdot\frac{\partial v_r^*}{\partial x^*} + \frac{\partial^2 v_r^*}{\partial x^*\partial r^*} + \frac{\partial^2 v_x^*}{\partial x^{*2}} = \left(\frac{c_s}{c_e}\right)^2\frac{\partial^2 v_x^*}{\partial t^{*2}}. \tag{12}$$

Let us investigate the case of infinite conductivity $\sigma = \infty$. The system (9) - (12) may then be reduced to one equation of the following type

$$\left[(1+\bar{m})\left(\frac{1}{c_0^{*2}}\cdot\frac{\partial^2}{\partial t^{*2}}-\frac{\partial^2}{\partial x^{*2}}\right)+\frac{\partial^2}{\partial x^{*2}}\right]\frac{\partial}{\partial r^*}\cdot\frac{1}{r^*}\cdot\frac{\partial}{\partial r^*}(r^*v_r^*)+$$
$$+\bar{m}\frac{\partial^2}{\partial x^{*2}}\left(\frac{1}{c_0^{*2}}\cdot\frac{\partial^2}{\partial t^{*2}}-\frac{\partial^2}{\partial x^{*2}}\right)v_r^*-\frac{1}{c_0^{*2}}\cdot\frac{\partial^2}{\partial t^{*2}}\left(\frac{1}{c_0^{*2}}\cdot\frac{\partial^2}{\partial t^{*2}}-\frac{\partial^2}{\partial x^{*2}}\right)v_r^*=0. \quad (13)$$

Representing all the desired functions in the following form

$$f(x^*, r^*, t^*) = F(r^*)e^{i(sx^*-\omega t^*)}, \quad (14)$$

we obtain the following from equation (13)

$$\frac{d^2V}{dr^{*2}}+\left(\frac{1}{r^*}-A\right)\frac{dV}{dr^*}+\left(B-\frac{1}{r^{*2}}\right)V=0, \quad (15)$$

where /228

$$B=\frac{\left[-\bar{m}s^2+\left(\frac{\omega}{c_0^*}\right)^2\right]\left[\left(\frac{\omega}{s}\right)^2-c_0^{*2}\right]}{\left(\frac{\omega}{s}\right)^2+\bar{m}\left[\left(\frac{\omega}{s}\right)^2-c_0^{*2}\right]} \quad (16)$$

The problem under consideration may be obtained as a particular case for A = 0.

By the following substitution (Ref. 3)

$$V = r^{*-\frac{1}{2}}e^{\frac{1}{2}Ar^*}u(k, m, \rho r^*), \quad (17)$$

where

$$\rho^2 = A^2-4B^2;\quad m=1;\quad k=\frac{A}{2\rho},$$

equation (15) may be reduced to a degenerate hypergeometric Whittaker equation

$$4(\rho r^*)^2\frac{d^2u}{dr^{*2}}=\left[(\rho r^*)^2-4\frac{A}{2\rho}(\rho r^*)+4m^2-1\right]u. \quad (18)$$

The solution of equation (18) may be written in the form of the combined Whittaker functions

$$u = C_1M_{k,m}(\rho r^*)+C_2M_{k,-m}(\rho r^*). \quad (19)$$

It may be assumed that the constant C_2 equals zero based on the condition that the solution is regular in the case $r^* \to 0$. Thus, the expression for radial velocity has the following form

$$V = C_1r^{*-\frac{1}{2}}e^{\frac{1}{2}Ar^*}M_{k,1}(\rho r^*). \quad (20)$$

The pressure derivative with respect to time may be determined from (5) and (6):

$$\frac{\partial p^*}{\partial t^*}=-c_0^{*2}\frac{\rho_0}{\gamma}\left[\frac{1}{r^*}\cdot\frac{\partial}{\partial r^*}(r^*v_r^*)+\frac{\partial v_x^*}{\partial x^*}\right]. \quad (21)$$

Determining the velocity v_x^* from (12), substituting in (21), and representing the solution in the form (14), we obtain

$$i\omega P(r^*)=c_0^{*2}\frac{\rho_0}{\lambda}\cdot\frac{\left(\frac{\omega}{s}\right)^2}{\left(\frac{\omega}{s}\right)^2-c_0^{*2}}\left(\frac{1}{r^*}V+\frac{dV}{dr^*}\right). \quad (22)$$

Utilizing the following relationships (Ref. 4)

$$\frac{d}{dz}M_{k,m}(z) = -\left(\frac{k}{2m+1} - \frac{2m+1}{2z}\right)M_{k,m}(z) + + \frac{(2m+1)^2 - 4k^2}{8(m+1)(2m+1)^2}M_{k,m+1}(z); \tag{23}$$

$$\frac{(2m+1)^2 - 4k^2}{16m(m+1)(2m+1)}M_{k,m+1}(z) = -\frac{4m^2 - 2kz - 1}{(2m-1)z}M_{k,m}(z) + + (2m+1)M_{k,m-1}(z), \tag{24}$$

we obtain the following from (22) /229

$$i\omega P(r^*) = c_0^{*3}\frac{\rho_0}{\gamma}\frac{\left(\frac{c}{c_0}\right)^2}{\left(\frac{c}{c_0}\right)^2 - 1}\left[\left(\frac{1}{2}A + k\rho\right)M_{k,1}(\rho r^*) + + 2\rho M_{k,0}(\rho r^*)\right]r^{*-\frac{1}{2}}e^{\frac{1}{2}Ar^*}C_1. \tag{25}$$

The axisymmetrical problem for an elastic medium with a cylindrical cavity may be reduced to solving two wave equations (Ref. 1):

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial x^2} = \frac{1}{c_e^2}\frac{\partial^2\varphi}{\partial t^2}; \tag{26}$$

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial\psi}{\partial r} - \frac{\psi}{r^2} + \frac{\partial^2\psi}{\partial x^2} = \frac{1}{c_s^2}\cdot\frac{\partial^2\psi}{\partial t^2} \tag{27}$$

for given boundary conditions on the cavity wall. The displacements and stresses may be expressed in terms of the functions ϕ and ψ by means of the well known formulas (Ref. 1). The condition for the absence of shearing stresses on the cavity wall is:

$$\tau = 0 \text{ for } r = r_0. \tag{28}$$

The second boundary condition expresses the relationship between elastic and conductive media and follows from the fact that the dynamic rigidities of the two media are equal on the boundary dividing them:

$$\frac{-\frac{\partial p}{\partial t}}{v_r} = \frac{\sigma_r}{R} \quad \text{or } r = r_0. \tag{29}$$

The solutions of equations (26) and (27) is written in terms of Macdonald functions (Ref. 1), and may be substituted in the expressions for radial displacement R and radial stress σ_r, with allowance for the boundary condition (28). The final formula in dimensionless quantities has the following form

$$\frac{\sigma_r^*}{R^*} = s\left\{\frac{(2-c^{*2})^2}{c^{*2}\left[1-\left(\frac{c_s}{c_e}\right)^2c^{*2}\right]^{\frac{1}{2}}}\cdot\frac{K_0\left(s\left[1-\left(\frac{c_s}{c_e}\right)^2c^{*2}\right]^{\frac{1}{2}}\right)}{K_1\left(s\left[1-\left(\frac{c_s}{c_e}\right)^2c^{*2}\right]^{\frac{1}{2}}\right)} + \right. + \frac{2}{c^{*2}}\cdot\frac{(2-c^{*2})\left[1-\left(\frac{c_s}{c_e}\right)^2c^{*2}\right]^{\frac{1}{2}}}{s\left[1-\left(\frac{c_s}{c_e}\right)^2c^{*2}\right]^{\frac{1}{2}}} - \left. -\frac{4}{c^{*2}}(1-c^{*2})\frac{1}{2}\left[\frac{1}{s(1-c^{*2})^{\frac{1}{2}}} + \frac{K_0(s[1-c^{*2}]^{\frac{1}{2}})}{K_1(s[1-c^{*2}]^{\frac{1}{2}})}\right]\right\}. \tag{30}$$

After substitution of the dimensionless quantities (A) and the dependences /230
(14), condition (29) assumes the following form

$$-\left.\frac{i\omega P(r^*)}{V(r^*)}\right|_{r^*=1} = c_0^*\left.\frac{\sigma_r^*(r^*)}{R^*(r^*)}\right|_{r^*=1}. \tag{31}$$

Substituting the expressions (20), (25) and (30) in (31), we obtain the dispersion equation connecting the phase velocity and the wavelength:

$$\begin{aligned}
&4\left[1-\left(\frac{c_0}{c_s}\right)^2\left(\frac{c}{c_0}\right)^2\right]^{\frac{1}{2}}\left\{\frac{1}{\frac{2\pi}{l^*}\left[1-\left(\frac{c_0}{c_s}\right)^2\left(\frac{c}{c_0}\right)^2\right]^{\frac{1}{2}}}+\frac{K_0\left(\frac{2\pi}{l^*}\left[1-\left(\frac{c_0}{c_s}\right)^2\left(\frac{c}{c_0}\right)^2\right]^{\frac{1}{2}}\right)}{K_1\left(\frac{2\pi}{l^*}\left[1-\left(\frac{c_0}{c_s}\right)^2\left(\frac{c}{c_0}\right)^2\right]^{\frac{1}{2}}\right)}\right\}- \\
&-\frac{2\left[2-\left(\frac{c_0}{c_s}\right)^2\left(\frac{c}{c_0}\right)^2\right]\left[1-\left(\frac{c_0}{c_e}\right)^2\left(\frac{c}{c_0}\right)^2\right]^{\frac{1}{2}}}{\frac{2\pi}{l^*}\left[1-\left(\frac{c_0}{c_s}\right)^2\left(\frac{c}{c_0}\right)^2\right]^{\frac{1}{2}}}-\frac{\left[2-\left(\frac{c_0}{c_s}\right)^2\left(\frac{c}{c_0}\right)^2\right]^2}{\left[1-\left(\frac{c_0}{c_s}\right)^2\left(\frac{c}{c_0}\right)^2\right]^{\frac{1}{2}}}\times \\
&\times\frac{K_0\left(\frac{2\pi}{l^*}\left[1-\left(\frac{c_0}{c_e}\right)^2\left(\frac{c}{c_0}\right)^2\right]^{\frac{1}{2}}\right)}{K_1\left(\frac{2\pi}{l^*}\left[1-\left(\frac{c_0}{c_e}\right)^2\left(\frac{c}{c_0}\right)^2\right]^{\frac{1}{2}}\right)}= \\
&=c_0^{*2}\frac{\rho_0}{\gamma}\cdot\frac{c^{*2}}{\frac{2\pi}{l^*}}\cdot\frac{\left(\frac{c}{c_0}\right)^2}{\left(\frac{c}{c_0}\right)^2-1}\left\{\frac{1}{2}A+k\rho+2\rho\frac{M_{k,0}(\rho)}{M_{k,1}(\rho)}\right\}.
\end{aligned} \tag{32}$$

The dispersion properties of the system under consideration may be determined by the following four characteristic parameters: c_0^*, ν, $\frac{\rho_0}{\gamma}$, $\bar{m}$.

If we set $\bar{m} = 0$ (there is no magnetic field), then the Whittaker functions degenerate into Bessel functions, and $A = k = 0$, and equation (32) describes wave dispersion in the case of a nonconductive liquid (Ref. 1). Let us study equation (32) in the two limiting cases of long and short waves.

In the case of long waves, the arguments of the Macdonald and Whittaker functions are small, and the expansion of these functions may be employed for small z (Ref. 6):

$$K_0(z)\approx\ln\frac{2}{z};\quad K_n(z)\approx\frac{1}{2}(n-1)!\left(\frac{z}{2}\right)^{-n};$$

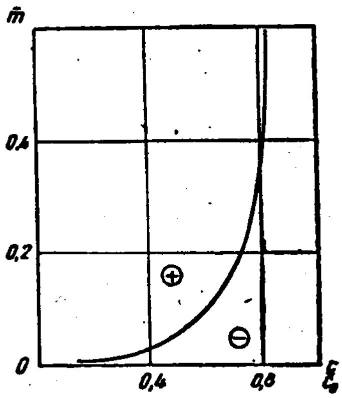
$$M_{k,m}(z)=z^{\frac{1}{2}+m}e^{-\frac{z}{2}}\left\{1+\frac{\frac{1}{2}+m-k}{1+2m}z+\frac{\left(\frac{1}{2}+m-k\right)\left(\frac{3}{2}+m-k\right)}{(1+2m)(2+2m)}\cdot\frac{z^2}{2}+\ldots\right\}.$$

In this case, equation (32) has one root corresponding to waves propagated /231
at a velocity which is smaller than the speed of sound in a nonconductive liquid. The approximate velocity may be determined by the following expression

$$\left(\frac{c}{c_0}\right)^2\approx\frac{1}{1+\left(\frac{c_0}{c_s}\right)^2\frac{\rho_0}{\gamma}}+\varepsilon M, \tag{33}$$

where the small parameter $\varepsilon=\left(\frac{2\pi}{\ell^*}\right)^2$ is introduced and

$$M=\frac{\frac{1}{2}c_0^{*2}\frac{\rho_0}{\gamma}\left(\frac{c}{c_0}\right)^2\left[1-\left(\frac{c}{c_0}\right)^2\right]}{\left(1+c_0^{*2}\frac{\rho_0}{\gamma}\right)\left\{-\left(\frac{c}{c_0}\right)^2+\bar{m}\left[1-\left(\frac{c}{c_0}\right)^2\right]\right\}^2}\left[-\frac{13}{2}(1+\bar{m})\left(\frac{c}{c_0}\right)^4+\right.$$
$$\left.+\left(\frac{49}{6}\bar{m}-\frac{5}{3}\bar{m}^2\right)\left(\frac{c}{c_0}\right)^2+\frac{5}{3}\bar{m}^2\right]. \tag{34}$$



According to formula (33), the value of $\frac{c}{c_0}$ may be determined by the method of successive approximations. The zero approximation holds in the case $\varepsilon = 0$. Substituting this value in the right hand side, we may determine the first approximation, etc.

Let us study the sign of M. It may be determined by the sign of the expression in the brackets. The value of $\bar{m}$, at which $M = 0$, may be determined by the following formula under the condition $\left(\frac{c}{c_0}\right)^2 \neq \frac{\bar{m}}{1+\bar{m}}$:

$$\bar{m}=\frac{\left(\frac{c}{c_0}\right)^2}{\frac{10}{3}\left[1-\left(\frac{c}{c_0}\right)^2\right]}\left\{-\left[\frac{49}{6}-\frac{13}{2}\left(\frac{c}{c_0}\right)^2\right]+\right.$$
$$\left.+\sqrt{\left[\frac{49}{6}-\frac{13}{2}\left(\frac{c}{c_0}\right)^2\right]^2+\frac{13}{2}}\right\}. \tag{35}$$

The figure shows the dependence of $\bar{m}$ on $\frac{c}{c_0}$, corresponding to formula (35). It may be seen that for small $\bar{m}$ the quantity $M < 0$ and the magnetic field decrease the phase velocity (the region located under the curve), and for large $\bar{m}$, $M > 0$, and the magnetic field increases the phase velocity (the region located above the curve).

In the case of short waves, the arguments of the Whittaker and Macdonald functions are larger, and the asymptotic representations of these functions may /232
be employed (Ref. 2, 6):

$$K_0(z)=\sqrt{\frac{\pi}{2z}}e^{-z}\left(1-\frac{1}{8z}+\cdots\right);\; K_1(z)=\sqrt{\frac{\pi}{2z}}e^{-z}\left(1+\frac{3}{8z}+\cdots\right); \tag{36}$$

$$M_{k,m}(z)=z^{-k}e^{\frac{z}{2}}\frac{\Gamma(2m+1)}{\Gamma\left(\frac{1}{2}-k+m\right)}\left\{1+\frac{\left(k+\frac{1}{2}\right)^2-m^2}{z}+\cdots\right\}. \tag{37}$$

When there is no magnetic field ($m = 0$), formula (32) may be reduced to an equation describing Rayleigh wave propagation. If we confine ourselves to the case $c < c_s$, we may obtain the following approximate relationship from the dispersion equation (32) by means of formulas (36) and (37)

$$\left(\frac{c}{c_0}\right)^2 \approx \frac{\bar{m}}{1+\bar{m}}\left(\frac{c_e}{c_s}\right)^2 \frac{\rho_0}{\gamma} \cdot \frac{\pi}{l^*}. \tag{38}$$

Formula (38) has a more formal meaning, since the magnetohydrodynamic model is not employed for short waves.

If the cavity wall is absolutely rigid, then the displacement $R^* \to 0$, and therefore $V(r^*)\big|_{r^*=1} \to 0$, from which we have the lower root

$$\left(\frac{c}{c_0}\right)^2 = \frac{\bar{m}}{1+\bar{m}}. \tag{39}$$

For small $\bar{m}$, we obtain the Alfvén wave velocity $c = a$; for large $\bar{m}$ we obtain the speed of sound $c = c_0$. In the case of arbitrary momentum, the solution may found by superposition of waves having the form (14).

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A CASE OF DYNAMIC STRESSES IN AN UNBOUNDED ELASTIC SPACE WITH A CYLINDRICAL CAVITY /233

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N67-24531

This article investigates the dynamic problem of stresses in unbounded space with a circular cylindrical cavity having the radius $r_0 = 1$. These stresses are produced due to the influence of force sources which change harmonically with time and which are uniformly distributed in two planes which are parallel to the cylinder axis $x = \pm \xi (\xi > 1)$. In particular, the stress state around the cavity is analyzed.

In an unbounded elastic medium with a cylindrical cavity having the radius $r_0 = 1$, let plane force sources be in operation in the case $x = \pm \xi (\xi > 1)$. The strength of these sources changes according to a harmonic law. Plane elastic waves are produced, due to the influence of the force sources in the medium. The cylindrical cavity will disturb the fundamental stress field which is produced due to propagation of elastic waves.

Fundamental stress field. In the case of the given system of sources, the displacement and stress will only depend on the variables x and t.

The equation for determining the displacements u is (Ref. 5)

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{c^2} q(x, t) = 0, \tag{1}$$

where c is the velocity of elastic wave propagation,

$$q(x, t) = \frac{q_0}{\rho}\{\delta(x-\xi) - \delta(x+\xi)\} e^{i\omega t}. \tag{2}$$

Here q_0 is a constant characterizing the strength of the force sources; ω -- oscillation frequency; δ -- Dirac function (Ref. 2).

We may select the solution of equation (1) in the following form /234

$$u(x, t) = u_0(x) e^{i\omega t}. \tag{3}$$

For $u_0(x)$, we then obtain the following value

$$u_0(x) = -\frac{q_0}{\rho\omega c} e^{-ik\xi} \sin k\xi; \quad x < \xi, \quad k = \frac{\omega}{c}. \tag{4}$$

The fundamental stress is

$$\sigma_x = \tilde{\sigma}_x e^{i\omega t} = A(\omega) e^{i\omega t} \cos kx, \tag{5}$$

where

$$A(\omega) = -\frac{\lambda + 2\mu}{\rho c^2} q_0 e^{-ik\xi} = -q_0 e^{-ik\xi}. \tag{6}$$

Disturbance stresses. The additional stresses which are produced due to the disturbance of the fundamental state by the cavity, may be conveniently determined by means of the potentials of longitudinal and transverse waves ϕ and ψ, which satisfy equations having the following form (Ref. 3)

$$\nabla^2 f_\nu - \frac{1}{c^2} \cdot \frac{\partial^2 f_\nu}{dt^2} = 0 \quad (\nu = 1,\ 2), \tag{7}$$

where

$$f_1 = \varphi, \quad f_2 = \psi,$$
$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_2 = \sqrt{\frac{\mu}{\rho}}.$$

In our case, we have

$$f_\nu = \tilde{f}_\nu e^{i\omega t}. \tag{8}$$

Conditions on the cavity surface will be

$$\left.\begin{aligned} \tilde{\sigma}_r^{(1)} + \tilde{\sigma}_r^{(2)} = 0; \\ \tilde{\tau}_{r\theta}^{(1)} + \tilde{\tau}_{r\theta}^{(2)} = 0, \end{aligned}\right\} r = 1 \tag{9}$$

and conditions of radiation, correspondingly will be

$$\lim_{r\to 0} \tilde{f}_\nu = 0; \quad \lim_{r\to 0} \sqrt{r}\left(\frac{\partial f_\nu}{\partial r} + ikf_\nu\right) = 0. \tag{10}$$

The upper index 1 in (9) pertains to the fundamental stress state, and 2 pertains to the disturbed stress state.

The fundamental stress state may be determined by expression (5). We may determine the disturbances according to formulas given (Ref. 3), which assume the following form with allowance for periodicity of the phenomenon and equation (7):

$$\left.\begin{aligned} \tilde{\sigma}_r^{(2)} &= -\lambda \frac{\omega^2}{c_1^2} \tilde{\varphi} + 2\mu \left(\frac{\partial^2 \tilde{\varphi}}{\partial r^2} - \frac{1}{r^2} \cdot \frac{\partial \tilde{\psi}}{\partial \theta} + \frac{1}{r} \cdot \frac{\partial^2 \tilde{\psi}}{\partial r \partial \theta} \right); \\ \tilde{\tau}_{r\theta}^{(2)} &= -\mu \frac{\omega^2}{c_2^2} \tilde{\psi} + 2\mu \left(\frac{1}{r} \cdot \frac{\partial^2 \tilde{\varphi}}{\partial r \partial \theta} - \frac{1}{r^2} \cdot \frac{\partial \tilde{\varphi}}{\partial \theta} - \frac{\partial^2 \tilde{\psi}}{\partial r^2} \right); \\ \tilde{\sigma}_\theta^{(2)} &= -2(\lambda + \mu) \frac{\omega^2}{c_1^2} \tilde{\varphi} - \tilde{\sigma}_r^{(2)}. \end{aligned}\right\} \tag{11}$$

Correspondingly, the amplitudes of the stress tensor components for the fundamental state in a polar coordinate system are (Ref. 4) /235

$$\left.\begin{aligned} \tilde{\sigma}_r^{(1)} &= \frac{1}{2} \tilde{\sigma}_x (1 + \cos 2\theta); \\ \tilde{\tau}_{r\theta}^{(1)} &= -\frac{1}{2} \tilde{\sigma}_x \sin 2\theta; \\ \tilde{\sigma}_\theta^{(1)} &= \frac{1}{2} \tilde{\sigma}_x (1 - \cos 2\theta). \end{aligned}\right\} \tag{12}$$

Let us employ the following series for cos kx (Ref. 1):

$$\cos kx = \sum_{n=0}^{\infty} a_{2n}^{*}(kr) \cos 2n\theta, \tag{13}$$

where

$$a_0^{*}(kr) = J_0(kr), \quad a_{2n}^{*}(kr) = (-1)^n J_{2n}(kr), \tag{14}$$

and $J_{2n}(kr)$ is the Bessel function of the first kind. Knowing (5), we have

$$\tilde{\sigma}_x = A(\omega) \sum_{n=0}^{\infty} a_{2n}^{*}(kr) \cos 2n\theta. \tag{15}$$

Then (12) assume the following form

$$\tilde{\sigma}_r^{(1)} = \frac{1}{2} A(\omega) \sum_{n=0}^{\infty} a_{2n}(kr) \cos 2n\theta; \tag{16}$$

$$\tilde{\tau}_{r\theta}^{(1)} = -\frac{1}{2} A(\omega) \sum_{n=0}^{\infty} b_{2n}(kr) \sin 2n\theta, \tag{16}$$

where

$$\left.\begin{aligned} a_0(kr) &= J_0(kr) - J_{2n}(kr); \\ a_{2n}(kr) &= (-1)^{n-1}\{J_{2n-2}(kr) - J_{2n}(kr) + J_{2n+2}(kr)\}; \\ b_{2n}(kr) &= (-1)^{n-1}\{J_{2n-2}(kr) - J_{2n+2}(kr)\}, \end{aligned}\right\} \tag{17}$$

Conditions (9) may be written as follows:

$$\begin{gathered} \tilde{\sigma}_r^{(2)} + \frac{1}{2} A(\omega) \sum_{n=0}^{\infty} a_{2n}(k) \cos 2n\theta = 0; \\ \tilde{\tau}_{r\theta}^{(2)} - \frac{1}{2} A(\omega) \sum_{n=1}^{\infty} b_{2n}(k) \sin 2n\theta = 0. \end{gathered} \tag{18}$$

Taking (18) into account, we shall try to find the solution of equations (7) in the form of the following series:

$$\begin{gathered} \tilde{\varphi}_n = \sum_{n=0}^{\infty} \varphi_n \cos 2n\theta; \\ \tilde{\psi}_n = \sum_{n=1}^{\infty} \psi_n \sin 2n\theta. \end{gathered} \tag{19}$$

With allowance for (7), we then obtain the following equations for determining the functions ϕ_n and ψ_n: /236

$$\begin{gathered} \frac{d^2 f_\nu}{dr^2} + \frac{1}{r} \cdot \frac{df_\nu}{dr} + \left(\frac{\omega^2}{c_1^2} - \frac{4n^2}{r^2}\right) f_\nu = 0 \quad (\nu = 1, 2); \\ f_1 = \varphi_n; \quad f_2 = \psi_n. \end{gathered} \tag{20}$$

Taking (11) into consideration, we obtain conditions (18) in the following form:

$$\begin{gathered} -\lambda \frac{\omega^2}{c_1^2} \varphi_n + 2\mu\left(\frac{d^2\varphi_n}{dr^2} - \frac{2n}{r^2}\psi_n + \frac{2n}{r} \cdot \frac{d\psi_n}{dr}\right) + \\ + \frac{1}{2} A(\omega) a_{2n}(kr) = 0; \\ \mu \frac{\omega^2}{c_2^2} \psi_n + 2\mu\left(\frac{2n}{r}\frac{d\varphi_n}{dr} - \frac{2n}{r^2}\varphi_n\right) + \frac{1}{2} A(\omega) b_{2n}(kr) = 0. \end{gathered} \tag{21}$$

We shall select the solutions of this system, satisfying the conditions of the problem, in the form of Hankel functions of the second kind (Ref. 1)

$$\varphi_n = A_n(\omega) H_{2n}^{(2)}\left(\frac{\omega}{c_1} r\right); \quad \psi_n = B_n(\omega) H_{2n}^{(2)}\left(\frac{\omega}{c_2} r\right). \tag{22}$$

We may determine the constants from (21) in the following form:

$$\begin{aligned} A_n(\omega) &= -\frac{1}{2} A(\omega) \frac{a_{2n}\left(\frac{\omega}{c_1}\right) F_{n4}(\omega) - b_{2n}\left(\frac{\omega}{c_1}\right) F_{n2}(\omega)}{F_{n1}(\omega) F_{n4}(\omega) - F_{n2}(\omega) F_{n3}(\omega)}; \\ B_n(\omega) &= -\frac{1}{2} A(\omega) \frac{b_{2n}\left(\frac{\omega}{c_1}\right) F_{n1}(\omega) - a_{2n}\left(\frac{\omega}{c_1}\right) F_{n3}(\omega)}{F_{n1}(\omega) F_{n4}(\omega) - F_{n2}(\omega) F_{n3}(\omega)}, \end{aligned} \tag{23}$$

where

$$\left.\begin{aligned} F_{n1}(\omega) &= \left\{-\lambda H_{2n}^{(2)}\left(\frac{\omega}{c_1}\right) + 2\mu H_{2n}^{(2)''}\left(\frac{\omega}{c_1}\right)\right\} \frac{\omega^2}{c_1^2}; \\ F_{n2}(\omega) &= 4\mu n \left\{\frac{\omega^2}{c_2^2} H_{2n}^{(2)}\left(\frac{\omega}{c_2}\right) - H_{2n}^{(2)}\left(\frac{\omega}{c_2}\right)\right\}; \end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned}F_{n3}(\omega) &= 4\mu n\left\{-H_{2n}^{(2)}\left(\frac{\omega}{c_1}\right)+\frac{\omega}{c_1}H_{2n}^{(2)\prime}\left(\frac{\omega}{c_1}\right)\right\};\\ F_{n4}(\omega) &= \mu\frac{\omega^2}{c_2^2}\left\{H_{2n}^{(2)}\left(\frac{\omega}{c_2}\right)+2H_{2n}^{(2)\prime\prime}\left(\frac{\omega}{c_2}\right)\right\}.\end{aligned}\right\} \quad (24)$$

Then, substituting (23) consecutively in (22), (19) and (21), we may determine the stresses σ_θ, σ_r, $\tau_{r\theta}$.

Stresses on the cavity surface. The stress amplitudes on the cavity surface may be determined according to the following formula:

$$\tilde{\sigma}_\theta = \tilde{\sigma}_\theta^{(1)} + \tilde{\sigma}_\theta^{(2)} \quad (r = 1). \quad (25)$$

However, we have /237

$$\tilde{\sigma}_\theta^{(2)} + \tilde{\sigma}_r^{(2)} = 2(\lambda+\mu)\nabla^2\tilde{\varphi} = -2(\lambda+\mu)\frac{\omega^2}{c_1^2}\tilde{\varphi},$$

and then

$$\tilde{\sigma}_\theta = -2(\lambda+\mu)\frac{\omega^2}{c_1^2}\tilde{\varphi} + \tilde{\sigma}_\theta^{(1)} - \tilde{\sigma}_r^{(2)} =$$
$$= -2(\lambda+\mu)\frac{\omega^2}{c_1^2}\tilde{\varphi} + (\tilde{\sigma}_\theta^{(1)} + \tilde{\sigma}_r^{(1)}) - (\tilde{\sigma}_r^{(1)} + \tilde{\sigma}_r^{(2)})$$

or, taking (12) into account, we obtain

$$\sigma_\theta = -2(\lambda+\mu)\frac{\omega^2}{c_1^2}\tilde{\varphi} + \tilde{\sigma}_x - \tilde{\sigma}_r. \quad (26)$$

On the cavity surface $\sigma_r = 0$, and consequently,

$$\tilde{\sigma}_\theta = -2(\lambda+\mu)\frac{\omega^2}{c_1^2}\tilde{\varphi} + \tilde{\sigma}_x \quad (r = 1). \quad (27)$$

Substituting (15) and (19) and taking into account (22), we obtain

$$\sigma_\theta = e^{i\omega t}\sum_{n=0}^{\infty} d_{2n}(\omega)\cos 2n\theta, \quad (28)$$

where

$$d_{2n}(\omega) = -2(\lambda+\mu)\frac{\omega^2}{c_1^2}A_n(\omega)H_{2n}^{(2)}\left(\frac{\omega}{c_1}\right) + A(\omega)a_{2n}^*(k), \quad (29)$$

and we may determine $a_{2n}^*(k)$ and $A_n(\omega)$ from (14) and (23), respectively.

Case of long waves. If the wavelength is large as compared with the cavity size -- i.e., $k = \frac{\omega}{c} \ll 1$ -- when calculating the stress we may employ expansion of Bessel functions in series in powers of the small argument (Ref. 1).

Retaining terms of the second order of smallness with respect to k, we obtain the following expression for stress on the cavity surface:

$$\begin{aligned}\operatorname{Re}(\sigma_\theta) &= -q_0(1-2\cos 2\theta)\cos\omega\left(t-\frac{\xi}{c}\right)+\\ &+q_0\left\{(\Phi_0-\Phi_1\cos 2\theta)\cos\omega\left(t-\frac{\xi}{c}\right)-\right.\\ &\left.-\left(\frac{\lambda+\mu}{\mu}\pi-\Phi_2\cos 2\theta\right)\sin\omega\left(t-\frac{\xi}{c}\right)\right\}\frac{\omega^2}{c_1^2},\end{aligned} \quad (30)$$

where /238

$$\left.\begin{aligned}
\Phi_0 &= \frac{1}{4} + \frac{\lambda+2\mu}{2\mu}\left(\ln\frac{\omega}{c_1} - \ln\frac{2}{\gamma}\right);\\
\Phi_1 &= \frac{1}{2}\,\frac{\mu}{\lambda+\mu}\left\{1 + \frac{\lambda}{\mu} + 2\left(1 + \frac{c_1^4}{c_2^4}\right)\right\}\left(\ln\frac{\omega}{c_2} - \ln\frac{2}{\gamma}\right);\\
\Phi_2 &= \frac{\mu}{\lambda+\mu}\left(1 + \frac{c_1^4}{c_2^4}\right);\quad \ln\frac{2}{\gamma} = 0{,}11593.
\end{aligned}\right\} \tag{31}$$

If $k \to 0$, we obtain the well known solution of G. Kirsch for the static case from (30).

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CONCENTRATION OF MOMENTS AT HOLES IN THE BENDING OF THIN PLATES WITH ALLOWANCE FOR PHYSICAL NONLINEARITY /239

N67-24532

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Hooke's law lies at the basis of classical (linear) elasticity theory, which considers small deformations. This law assumes a linear dependence between stresses and deformations. However, there is a nonlinear dependence between stresses and deformations in many materials employed in technology (polymers, alloys, non-ferrous metals, etc.), even in the region of small deformations. Thus, Hooke's law is replaced by a nonlinear elasticity law. If the geometric relationships of classical elasticity theory remain in force, we arrive at one of the variations of the general elasticity theory -- the so-called physically nonlinear law. The monograph by G. Kauderer (Ref. 2) presents the fundamental laws and hypotheses for physically nonlinear bodies. In this monograph, on the basis of experimental research the nonlinear elasticity law for the metals indicated above is represented in the following form:

$$T = 3K\chi(\varepsilon_0)D_0 + 2G\gamma(\psi_0^2)D', \tag{1}$$

where T is the stress tensor; D_0, D' -- the spherical tensor and the deformation deviator, respectively; $\chi(\varepsilon_0)$, $\gamma(\psi_0^2)$ -- extension and displacement functions whose behavior may be established experimentally, just like the moduli of shear G and the volumetric contraction K; ε_0, ψ_0 -- the average extension and intensity of the shear deformation, respectively, which may be expressed by the well known formulas

$$\varepsilon_0 = \frac{1}{3}(\varepsilon_x + \varepsilon_y + \varepsilon_z);$$

$$\psi_0 = \frac{2}{\sqrt{3}}\sqrt{\frac{2}{3}(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 - \varepsilon_x\varepsilon_y - \varepsilon_y\varepsilon_z - \varepsilon_z\varepsilon_x) + \frac{1}{2}(\psi_{xy}^2 + \psi_{yz}^2 + \psi_{zx}^2)}\,.$$

This article investigates the problem of the bending of thin plates for small deflection, whose material follows a nonlinear law of elasticity, and small deformations. A solution is provided for the problem of the moment concentration in a circular plate which is loaded axysymmetrically, and also in a plate which is weakened by a circular hole, under conditions of pure cylindrical bending. The influence of the external loading and the elastic properties of the material upon the moment concentration coefficient is investigated. /240

Fundamental bending equations of thin plates and method of solution. The article (Ref. 6) investigated the problem of the bending of thin plates for small deflections, whose material follows a nonlinear elasticity law (Ref. 2), and small deformations. In this study, we obtained the fundamental relationships for the bending of thin plates.

The equation for the bending of such plates in cylindrical coordinates (r, ϕ, z) has the following form

$$(1-\nu_0)\left\{F\Delta\Delta w+2F_r\frac{\partial}{\partial r}\Delta w+\frac{2}{r^2}F_\varphi\frac{\partial}{\partial\varphi}\Delta w+\left(F_{rr}+\frac{1}{r}\Phi_r+\right.\right.$$
$$\left.+\frac{1}{r^2}\Phi_{\varphi\varphi}\right)w_{rr}+\left(\frac{1}{r}F_r+\frac{1}{r^2}F_{\varphi\varphi}+\Phi_{rr}\right)\left(\frac{1}{r}w_r+\frac{1}{r^2}w_{\varphi\varphi}\right)+ \tag{2}$$
$$\left.+\frac{2}{r^2}\left[F_{r\varphi}-\Phi_{r\varphi}-\frac{1}{r}(F_\varphi-\Phi_\varphi)\right]\left(w_{r\varphi}-\frac{1}{r}w_\varphi\right)\right\}=\frac{q(r,\varphi)}{D},$$

where $w(r, \phi)$ is the deflection of points in the middle plane of the plate;

$q(r, \phi)$ -- continuous load which is perpendicular to the middle plane;

ν_0 -- Poisson coefficient; D -- cylindrical rigidity, which is related to the plate thickness h and the moduli K and G by the following formula

$$D=\frac{1}{3}\cdot\frac{3K+G}{3K+4G}Gh^3,$$

and Δ designates the Laplace operator.

The functions $F(r, \phi)$ and $\Phi(r, \phi)$ represent integrals of the following type

$$F(r,\varphi)=\frac{12}{h^3}\int\limits_{-\frac{h}{2}}^{+\frac{h}{2}}\frac{\gamma(\psi_0^2)}{1-\nu(\varepsilon_0,\psi_0^2)}z^2dz; \tag{3}$$

$$\Phi(r,\varphi)=\frac{12}{h^3}\int\limits_{-\frac{h}{2}}^{+\frac{h}{2}}\frac{\gamma(\psi_0^2)\,\nu(\varepsilon_0,\psi_0^2)}{1-\nu(\varepsilon_0,\psi_0^2)}z^2dz; \tag{4}$$
$$\nu(\varepsilon_0,\psi_0^2)=\frac{1}{2}\cdot\frac{3K\chi(\varepsilon_0)-2G\gamma(\psi_0^2)}{3K\chi(\varepsilon_0)+G\gamma(\psi_0^2)},$$

and the following formulas hold for average extension and the square of the shear deformation intensity /241

$$\varepsilon_0=-\frac{1}{3}\cdot\frac{1-2\nu_0}{1-\nu_0}z\Delta w;$$
$$\psi_0^2=\frac{8}{9}\left\{\nu_1\left[w_{rr}^2+\frac{1}{r^2}\left(w_r+\frac{1}{r}w_{\varphi\varphi}\right)^2\right]+\frac{\nu_2}{r}w_{rr}\left(w_r+\frac{1}{r}w_{\varphi\varphi}\right)+\right.$$
$$\left.+\frac{3}{r^2}\left(w_{r\varphi}-\frac{1}{r}w_\varphi\right)^2\right\}z^2,$$

where

$$\nu_1=\frac{\nu_0}{(1-\nu_0)^2}+1;\quad \nu_2=\frac{2\nu_0}{(1-\nu_0)^2}-1.$$

From this point, the subscripts indicate the derivatives with respect to r and ϕ of the functions obtained.

The following expressions are obtained for the moments M_r, M_ϕ and $M_{r\phi}$

$$\left.\begin{aligned}M_r&=-D(1-\nu_0)\left[Fw_{rr}+\Phi\left(\frac{1}{r}w_r+\frac{1}{r^2}w_{\varphi\varphi}\right)\right];\\M_\varphi&=-D(1-\nu_0)\left[F\left(\frac{1}{r}w_r+\frac{1}{r^2}w_{\varphi\varphi}\right)+\Phi w_{rr}\right];\\M_{r\varphi}&=-D(1-\nu_0)(F-\Phi)\left(\frac{1}{r}w_{r\varphi}-\frac{1}{r^2}w_\varphi\right).\end{aligned}\right\} \tag{5}$$

In order to obtain the expression for the intersection force -- for example, Q_r -- instead of M_r, M_ϕ and $M_{r\phi}$ we must substitute their values in the expression

$$Q_r = \frac{\partial M_r}{\partial r} + \frac{M_r}{r} - \frac{M_\varphi}{r} + \frac{1}{r} \cdot \frac{\partial M_{r\varphi}}{\partial \varphi}$$

and must perform the corresponding operations.

We shall confine ourselves to studying the case when there is a small deviation of the nonlinear elasticity law from Hooke's law. For many materials, the functions of extension and shear then have the following form

$$\chi(\varepsilon_0) \equiv 1; \; \gamma(\psi_0^2) = 1 - g_2\psi_0^2 \; (g_2\psi_0^2 \ll 1), \tag{6}$$

where g_2 is an elastic constant which may be determined experimentally. Taking (6) into account, we obtain the following expression for the function $\nu(\varepsilon_0, \psi_0^2)$:

$$\nu(\varepsilon_0, \psi_0^2) = \frac{1}{2} \cdot \frac{3K - 2G(1 - g_2\psi_0^2)}{3K + G(1 - g_2\psi_0^2)}.$$

Expanding the right hand side in powers of g_2, and taking into account the term which is linear with respect to g_2 in the first approximation ($g_2\psi_0^2 \ll 1$), we obtain

$$\nu(\varepsilon_0, \psi_0^2) = \nu_0 + \frac{1}{3}(1 + \nu_0)(1 - 2\nu_0) g_2\psi_0^2.$$

Let us introduce this value in the expressions $\dfrac{1}{1 - \nu(\varepsilon_0, \psi_0^2)}$ and $\dfrac{\nu(\varepsilon_0, \psi_0^2)}{1 - \nu(\varepsilon_0, \psi_0^2)}$ which are included in the integrals (3). Expanding these expressions in series and truncating them at terms which are linear with respect to g_2, we obtain /242

$$\frac{1}{1 - \nu(\varepsilon_0, \psi_0^2)} = \frac{1}{1 - \nu_0}\left[1 + \frac{1}{3} \cdot \frac{(1 + \nu_0)(1 - 2\nu_0)}{1 - \nu_0} g_2\psi_0^2\right];$$

$$\frac{\nu(\varepsilon_0, \psi_0^2)}{1 - \nu(\varepsilon_0, \psi_0^2)} = \frac{\nu_0}{1 - \nu_0}\left[1 + \frac{1}{3} \cdot \frac{(1 + \nu_0)(1 - 2\nu_0)}{\nu_0(1 - \nu_0)} g_2\psi_0^2\right].$$

Let us now calculate the integrals (3), expanding them in series and retaining only terms which are linear with respect to g_2 in these series. For the functions $F(r, \phi)$ and $\Phi(r, \phi)$, we then obtain the expressions

$$F(r, \varphi) = \frac{1}{1 - \nu_0} - \lambda s(r, \varphi);$$

$$\Phi(r, \varphi) = \frac{\nu_0}{1 - \nu_0} - \lambda t(r, \varphi),$$

where $\lambda = \dfrac{g_2 K}{(3K + G)\, G^2}$ is a small parameter which characterizes the deviation from the linear elasticity law; the functions $s(r, \phi)$ and $t(r, \phi)$ have the following form

$$s(r, \varphi) = \frac{72}{5} \cdot \frac{(1-\nu_0)^2 D^2}{(1+\nu_0) h^4} \nu_1 \left\{\nu_1 \left[w_{rr}^2 + \frac{1}{r^2}\left(w_r + \frac{1}{r} w_{\varphi\varphi}\right)^2\right] + \right.$$
$$\left. + \frac{\nu_2}{r} w_{rr}\left(w_r + \frac{1}{r} w_{\varphi\varphi}\right) + \frac{3}{r^2}\left(w_{r\varphi} - \frac{1}{r} w_\varphi\right)^2\right\};$$
$$t(r, \varphi) = \frac{36(1-\nu_0)^2 D^2}{5(1+\nu_0) h^4} \nu_2 \left\{\nu_1 \left[w_{rr}^2 + \frac{1}{r^2}\left(w_r + \frac{1}{r} w_{\varphi\varphi}\right)^2\right] + \right.$$
$$\left. + \frac{\nu_2}{r} w_{rr}\left(w_r + \frac{1}{r} w_{\varphi\varphi}\right) + \frac{3}{r^2}\left(w_{r\varphi} - \frac{1}{r} w_\varphi\right)^2\right\}.$$

Introducing these expressions in (2), we obtain the equation for the plate bending in the following form:

$$\Delta\Delta w - \lambda L[w] = \frac{q}{D}; \tag{7}$$

and thus $L[w]$ has the form

$$L[w] = (1-\nu_0)\left\{s\Delta\Delta w + 2s_r \frac{\partial}{\partial r}\Delta w + \frac{2}{r^2} s_\varphi \frac{\partial}{\partial \varphi}\Delta w + \right.$$
$$+ \left(s_{rr} + \frac{1}{r} t_r + \frac{1}{r^2} t_{\varphi\varphi}\right) w_{rr} + \left(\frac{1}{r} s_r + \frac{1}{r^2} s_{\varphi\varphi} + t_{rr}\right)\left(\frac{1}{r} w_r + \frac{1}{r^2} w_{\varphi\varphi}\right) + \tag{8}$$
$$\left. + \frac{2}{r^2}\left[s_{r\varphi} - t_{r\varphi} - \frac{1}{r}(s_\varphi - t_\varphi)\right]\left(w_{r\varphi} - \frac{1}{r} w_\varphi\right)\right\}.$$

The formulas for the bending moments (5) will have the following form /243

$$M_r = -D\left\{w_{rr} + \nu_0\left(\frac{1}{r} w_r + \frac{1}{r^2} w_{\varphi\varphi}\right) - \lambda\left[s w_{rr} + \right.\right.$$
$$\left.\left. + t\left(\frac{1}{r} w_r + \frac{1}{r^2} w_{\varphi\varphi}\right)\right](1-\nu_0)\right\},$$
$$M_\varphi = -D\left\{\nu_0 w_{rr} + \left(\frac{1}{r} w_r + \frac{1}{r^2} w_{\varphi\varphi}\right) - \lambda\left[s\left(\frac{1}{r} w_r + \frac{1}{r^2} w_{\varphi\varphi}\right) + \right.\right.$$
$$\left.\left. + t w_{rr}\right](1-\nu_0)\right\},$$
$$M_{r\varphi} = -D\left\{\frac{1}{r} w_{r\varphi} - \frac{1}{r^2} w_\varphi - \lambda\left[(s-t)\left(\frac{1}{r} w_{r\varphi} - \frac{1}{r^2} w_\varphi\right)\right]\right\}(1-\nu_0).$$

We shall try to find the solution of equation (7) by expanding it in powers of the small parameter λ which is contained in this equation (Ref. 1, 2, 4):

$$w(r, \varphi, \lambda) = w^{(0)}(r, \varphi) + \lambda w^{(1)}(r, \varphi) + \lambda^2 w^{(2)}(r, \varphi) + \ldots, \tag{9}$$

where $w^{(0)}(r, \phi)$, $w^{(1)}(r, \phi)$, ... are the functions of the zero, first, and higher approximations.

Substituting $w(r, \phi, \lambda)$ in (7) and setting the coefficients equal to zero for identical powers of λ, we obtain an infinite system of biharmonic equations for determining the functions $w^{(n)}(r, \phi)$ $(n = 0, 1, 2, \ldots)$.

We obtain the following equation for the function $w^{(0)}(r, \phi)$

$$\Delta\Delta w^{(0)} = \frac{q}{D}, \tag{10}$$

which corresponds to the bending of thin plates, for whose material Hooke's law is valid.

For the function of the first approximation $w^{(1)}$, we obtain the following equation

$$\Delta\Delta w^{(1)} = L\,[w^{(0)}]. \tag{11}$$

Let us confine ourselves to the approximations of the zero and first orders. The solution of the thin plate bending problem in the first approximations may be reduced to integrating equations (10), (11) under the corresponding boundary conditions.

Circular plate which is axisymmetrically loaded. Let us investigate a circular plate with a hinge-supported outer edge subjected to the moment $M_r = M$ which is distributed uniformly over the outer profile. We shall assume that the inner profile is free. Let us employ a and b to designate the outer and inner radii of the plate. The functions which may be employed in solving this problem do not depend on the coordinate ϕ.

The function $w^{(0)}$ may be found from equation (10) (in the case $q = 0$) for the following boundary conditions /244

$$w^{(0)} = 0;\ -D\left(w_{rr}^{(0)} + \frac{\nu_0}{r} w_r^{(0)}\right) = M \text{ for } r = a;$$

$$-D\left(w_{rr}^{(0)} + \frac{\nu_0}{r} w_r^{(0)}\right) = 0;\ -D\frac{d}{dr}\Delta w^{(0)} = 0 \text{ for } r = b.$$

According to the linear theory of thin plate bending, we have

$$w^{(0)} = -\frac{Ma^2b^2}{D(1-\nu_0)(a^2-b^2)}\ln\frac{r}{a} + \frac{Ma^2}{2D(1+\nu_0)(a^2-b^2)}(a^2-r^2).$$

If the function $w^{(0)}$ is known, in accordance with equation (11) we may write a differential equation for the first approximating function:

$$\Delta\Delta w^{(1)} = \frac{864a^6b^4M^3}{5(1+\nu_0)(a^2-b^2)^3Dh^4}\left(-2\frac{1+\nu_0}{1-\nu_0}\cdot\frac{1}{r^6} + 9\frac{b^2}{r^8}\right). \tag{12}$$

Thus, the following boundary conditions must be satisfied:

$$w^{(1)} = 0;\ w_{rr}^{(1)} + \frac{\nu_0}{r} w_r^{(1)} = (1-\nu_0)\left(s^{(0)}w_{rr}^{(0)} + t^{(0)}\frac{w_r^{(0)}}{r}\right) \text{ for } r = a$$

and

$$\left.\begin{aligned} & w_{rr}^{(1)} + \frac{\nu_0}{r} w_r^{(1)} = (1-\nu_0)\left(s^{(0)}w_{rr}^{(0)} + t^{(0)}\frac{w_r^{(0)}}{r}\right) \text{ for } r = b; \\ & \frac{d}{dr}\Delta w^{(1)} = (1-\nu_0)\left(s^{(0)}\frac{d}{dr}\Delta w^{(0)} + s_r^{(0)}w_{rr}^{(0)} + t_r^{(0)}\frac{w_r^{(0)}}{r}\right). \end{aligned}\right\} \tag{13}$$

We may represent the solution of equation (12) in the form of the sum of the particular solution of this equation and the general solution of the corresponding homogeneous equation $\Delta\Delta w^{(1)} = 0$. The integration constants of this equation may be found from the condition (13).

The function $w^{(1)}$ will thus have the following form:

$$w^{(1)} = R_1\ln\frac{r}{a} + R_2\left(1-\frac{r^2}{a^2}\right) + R_3\left(1-\frac{a^2}{r^2}\right) + R_4\left(1-\frac{a^4}{r^4}\right). \tag{14}$$

In order to calculate the bending moment M_ϕ in the first approximation, the following expression is obtained

$$M_\varphi = -D\Big\{\nu_0 w_{rr}^{(0)} + \Big(\frac{1}{r} w_r^{(0)} + \frac{1}{r^2} w_{\varphi\varphi}^{(0)}\Big) + \lambda\Big[\nu_0 w_{rr}^{(1)} + \Big(\frac{1}{r} w_r^{(1)} + \\ + \frac{1}{r^2} w_{\varphi\varphi}^{(1)}\Big)\Big] - \lambda\Big[s^{(0)}\Big(\frac{w_r^{(0)}}{r} + \frac{w_{\varphi\varphi}^{(0)}}{r^2}\Big) + t^{(0)} w_{rr}^{(0)}\Big](1-\nu_0)\Big\}.$$

As is known from the linear theory of thin plate bending (Ref. 5), $M_\phi >$ /245
$> M_r$, and reaches the largest value on the inner profile. Thus, the concentration coefficient is

$$\left(\frac{M_\varphi}{M}\right)_{r=b} = \frac{2k^2}{k^2-1}.$$

In our case, the expression for M_ϕ on the profile is as follows:

$$M_\varphi = \frac{2k^2 M}{k^2-1}\left\{1 - \frac{54\,[4k^4 - (k^2+1)^2]}{5\,(k^2-1)^2 h^4} M^2\lambda\right\}, \tag{15}$$

and the concentration coefficient

$$\left(\frac{M_\varphi}{M}\right)_{r=b} = \frac{2k^2}{k^2-1}\left\{1 - \frac{54\,[4k^4 - (k^2+1)^2]}{5\,(k^2-1)^2 h^4} M^2\lambda\right\}\ \left(k = \frac{a}{b}\right)$$

will depend on the external loading, the mechanical properties of the material, and the plate thickness.

Figure 1 presents a graph showing the change in the concentration coefficient of the moment M_ϕ over the inner profile as a function of the external moment M for separate materials in the case $k = \frac{3}{2}$, h = 1 cm. Curve 1 represents copper with the characteristics $K = 1.33\cdot10^7$ n/cm^2; $G = 0.47\cdot10^7$ n/cm^2; $g_2 =$ $= 7.26\cdot10^6$; $\lambda = 0.98\cdot10^{-7}$ cm^4/n^2. Curve 2 is for pure copper with the characteristics $K = 1.37\cdot10^7$ n/cm^2, $G = 0.46\cdot10^7$ n/cm^2; $g_2 = 0.18\cdot10^6$; $\lambda = 0.255\cdot10^{-8}$ cm^4/n^2.

As may be seen from the graph, the reduction in the moment concentration coefficient around the hole, which is frequently observed, may be explained by the behavior of the material which is nonlinearly elastic.

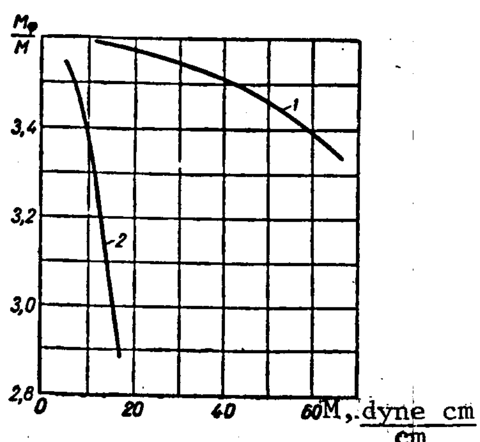

Figure 1

Pure cylindrical bending of a thin plate weakened by a circular hole. Let us investigate a rectangular thin plate which is weakened by a circular hole having the radius a, under conditions of pure cylindrical bending. We shall assume that the hole radius is small as compared with the plate dimensions (length and width), and the hole is so small that it has no influence upon the stress state over the external plate profile. Under these assumptions, this plate may be conditionally regarded as an infinite plate which is weakened by a circular hole.

The function $w^{(0)}$ may be found from equation (10), when q = 0 /246

$$\Delta\Delta w^{(0)} = 0,$$

under the following boundary conditions:

$$\left.\begin{aligned} &-D\left[w_{rr}^{(0)} + \nu_0\left(\frac{1}{r}w_r^{(0)} + \frac{1}{r^2}w_{\varphi\varphi}^{(0)}\right)\right] = 0;\\ &-D\left[\frac{\partial}{\partial r}\left(w_{rr}^{(0)} + \frac{1}{r}w_r^{(0)}\right) + \left(-\frac{3w_{\varphi\varphi}^{(0)}}{r^3} + \frac{2w_{r\varphi\varphi}^{(0)}}{r^2}\right) + \nu_0\left(-\frac{w_{r\varphi\varphi}^{(0)}}{r^2} + \frac{w_{\varphi\varphi}^{(0)}}{r^3}\right)\right] = 0 \end{aligned}\right\} \text{ for } r = a,$$

$$\left.\begin{aligned} &-D\left[w_{rr}^{(0)} + \nu_0\left(\frac{1}{r}w_r^{(0)} + \frac{1}{r^2}w_{\varphi\varphi}^{(0)}\right)\right] = M\cos^2\varphi;\\ &-D\left[\left(\frac{1}{r}w_r^{(0)} + \frac{1}{r^2}w_{\varphi\varphi}^{(0)}\right) + \nu_0 w_{rr}^{(0)}\right] = M\sin^2\varphi;\\ &-D\left(\frac{1}{r}w_{r\varphi}^{(0)} - \frac{1}{r^2}w_{\varphi}^{(0)}\right) = -\frac{1}{2}M\sin 2\varphi \end{aligned}\right\} \text{ for } r = \infty,$$

represents a solution of this problem in the linear formulation (Ref. 3), and may be written in the following form

$$w^{(0)}(r,\ \varphi) = C_1 r^2 + C_3\ln r + \left(C_2 r^2 + C_4 + \frac{C_5}{r^2}\right)\cos 2\varphi,$$

where

$$C_1 = -\frac{M}{4(1+\nu_0)D};\quad C_3 = -\frac{Ma^2}{2(1-\nu_0)D};$$

$$C_2 = -\frac{M}{4(1-\nu_0)D};\quad C_4 = -\frac{Ma^2}{2(3+\nu_0)D};\quad C_5 = \frac{Ma^4}{4(3+\nu_0)D}.$$

According to (11), we obtain the following differential equation for $w^{(1)}(r,\ \phi)$

$$\Delta\Delta w^{(1)} = \frac{A_1}{r^6} + \frac{A_2}{r^8} + \frac{A_3}{r^{10}} + \frac{A_4}{r^{12}} + \left(\frac{A_5}{r^6} + \frac{A_6}{r^8} + \frac{A_7}{r^{10}} + \frac{A_8}{r^{12}} + \frac{A_9}{r^{14}}\right)\cos 2\varphi + \left(\frac{A_{10}}{r^4} + \frac{A_{11}}{r^6} + \frac{A_{12}}{r^8} + \frac{A_{13}}{r^{10}} + \frac{A_{14}}{r^{12}}\right)\cos 4\varphi + \left(\frac{A_{15}}{r^4} + \frac{A_{16}}{r^6}\right)\cos 6\varphi, \tag{16}$$

where

$$A_1 = -\frac{216a^2c}{(1-\nu_0)(3+\nu_0)}(\nu_0^3 + 15\nu_0^2 + 63\nu_0 + 1);$$

$$A_2 = \frac{324a^4c}{3+\nu_0}(29\nu_0^2 - 2\nu_0 + 5);\quad A_3 = -\frac{288a^6c}{3+\nu_0}(13\nu_0^2 - 40\nu_0 + 19);$$

$$A_4 = \frac{58320(1-\nu_0)^2 a^8 c}{3+\nu_0};\quad A_5 = -\frac{1944a^2c}{3+\nu_0}(5\nu_0^2 - 2\nu_0 + 13);$$

$$A_6 = \frac{864a^4c}{(1-\nu_0)(3+\nu_0)^2}(30\nu_0^3 + 372\nu_0^2 - 350\nu_0 + 81);$$

$$A_7 = \frac{1296(1-\nu_0)a^6c}{(3+\nu_0)^2}(73\nu_0^2 + 86\nu_0 - 11);$$

$$A_8 = -\frac{11664(1-\nu_0)a^8c}{(3+\nu_0)^2}(7\nu_0^2 - 22\nu_0 + 7);\quad A_9 = \frac{17496(1-\nu_0)a^{10}}{(3+\nu_0)^2}\cdot 5c;$$ /247

$$A_{10} = \frac{324c}{1}(13 + 7\nu_0);\quad A_{11} = \frac{7776(1-\nu_0)}{1}a^2c;$$

$$A_{12} = -19440(1-\nu_0)a^4c;\quad A_{13} = \frac{15552\nu_0 a^6c}{3+\nu_0};$$

$$A_{14} = \frac{2916(1-\nu_0)^2 a^6 c}{3+\nu^0}; \quad A_{15} = -3888(1-\nu_0)c;$$
$$A_{16} = -\frac{216a^4c}{3+\nu_0}(61\nu_0^2 + 98\nu_0 - 119); \quad c = \frac{a^2M^3}{5(1+\nu_0)(3+\nu_0)h^4D}.$$

Thus, the following boundary conditions must be satisfied:

$$w_{rr}^{(1)} + \nu_0\left(\frac{1}{r}w_r^{(1)} + \frac{1}{r^2}w_{\varphi\varphi}^{(1)}\right) = (1-\nu_0)\left[s^{(0)}w_{rr}^{(0)} + t^{(0)}\left(\frac{1}{r}w_r^{(0)} + \frac{1}{r^2}w_{\varphi\varphi}^{(0)}\right)\right],$$

$$\frac{\partial}{\partial r}\left(w_{rr}^{(1)} + \frac{1}{r}w_r^{(1)}\right) + \left(-\frac{3w_{\varphi\varphi}^{(1)}}{r^3} + \frac{2w_{r\varphi\varphi}^{(1)}}{r^2}\right) + \nu_0\left(-\frac{w_{r\varphi\varphi}^{(1)}}{r^2} + \frac{w_{\varphi\varphi}^{(1)}}{r^3}\right) =$$
$$= (1-\nu_0)\left\{s^{(0)}\left[\frac{\partial}{\partial r}\left(w_{rr}^{(0)} + \frac{w_r^{(0)}}{r}\right) - \frac{3w_{\varphi\varphi}^{(0)}}{r^3} + \frac{2w_{r\varphi\varphi}^{(0)}}{r^2}\right] + \right.$$
$$+ t^{(0)}\left(-\frac{w_{r\varphi\varphi}^{(0)}}{r^2} + \frac{w_{\varphi\varphi}^{(0)}}{r^3}\right) + s_r^{(0)}w_{rr}^{(0)} + t_r^{(0)}\left(\frac{w_r^{(0)}}{r} + \frac{w_{\varphi\varphi}^{(0)}}{r^2}\right) +$$
$$\left. + 2(s_\varphi^{(0)} - t_\varphi^{(0)})\left(\frac{w_{r\varphi}^{(0)}}{r^2} - \frac{w_\varphi^{(0)}}{r^3}\right)\right\} \text{ for } r = a; \tag{17}$$

$$w_{rr}^{(1)} + \nu_0\left(\frac{1}{r}w_r^{(1)} + \frac{1}{r^2}w_{\varphi\varphi}^{(1)}\right) = (1-\nu_0)\left[s^{(0)}w_{rr}^{(0)} + t^{(0)}\left(\frac{1}{r}w_r^{(0)} + \frac{1}{r^2}w_{\varphi\varphi}^{(0)}\right)\right],$$

$$\left(\frac{1}{r}w_r^{(1)} + \frac{1}{r^2}w_{\varphi\varphi}^{(1)}\right) + \nu_0 w_{rr}^{(1)} = (1-\nu_0)\left[s^{(0)}\left(\frac{1}{r}w_r^{(0)} + \frac{1}{r^2}w_{\varphi\varphi}^{(0)}\right) + t^{(0)}w_{rr}^{(0)}\right]$$
$$\text{for} \quad r = \infty;$$
$$\frac{1}{r}w_{r\varphi}^{(1)} - \frac{1}{r^2}w_\varphi^{(1)} = (1-\nu_0)(s^{(0)} - t^{(0)})\left(\frac{1}{r}w_{r\varphi}^{(0)} - \frac{1}{r^2}w_\varphi^{(0)}\right).$$

Here the functions $s^{(0)}$ and $t^{(0)}$ equal the corresponding functions s and t, in which $w^{(0)}$ is substituted instead of w.

We may write the solution of equation (16) in the form of the sum of the particular solution for this equation and the general solution of the corresponding homogeneous equation $\Delta\Delta w^{(1)} = 0$. We may find the integration constants of this equation from condition (17).

The function $w^{(1)}$ may be expressed as follows

$$w^{(1)}(r, \varphi) = R_0 r^2 + R_1 \ln r + \frac{\alpha_1}{r^2} + \frac{\alpha_2}{r^4} + \frac{\alpha_3}{r^6} + \frac{\alpha_4}{r^8} +$$
$$+ \left(R_4 + R_2 r^2 + \frac{R_3}{r^2} + \alpha_5\frac{\ln r}{r^2} + \frac{\alpha_6}{r^4} + \frac{\alpha_7}{r^6} + \frac{\alpha_8}{r^8} + \frac{\alpha_9}{r^{10}}\right)\cos 2\varphi +$$
$$+ \left(\alpha_{10} + \alpha_{11}\frac{\ln r}{r^2} + \frac{R_6}{r^2} + \frac{R_5}{r^4} + \alpha_{12}\frac{\ln r}{r^4} + \frac{\alpha_{13}}{r^6} + \frac{\alpha_{14}}{r^8}\right)\cos 4\varphi +$$
$$+ \left(\alpha_{15} + \frac{\alpha_{16}}{r^2} + \frac{R_8}{r^4} + \frac{R_7}{r^6}\right)\cos 6\varphi, \ldots$$

where /248

$$\alpha_1 = \frac{A_1}{64}; \quad \alpha_2 = \frac{A_2}{576}; \quad \alpha_3 = \frac{A_3}{2304}; \quad \alpha_4 = \frac{A_4}{6400};$$
$$\alpha_5 = -\frac{A_5}{48}; \quad \alpha_6 = \frac{A_6}{384}; \quad \alpha_7 = \frac{A_7}{1920}; \quad \alpha_8 = \frac{A_8}{5760};$$
$$\alpha_9 = \frac{A_9}{13440}; \quad \alpha_{10} = \frac{A_{10}}{192}; \quad \alpha_{11} = \frac{A_{11}}{96}; \quad \alpha_{12} = -\frac{A_{12}}{160};$$
$$\alpha_{13} = \frac{A_{13}}{960}; \quad \alpha_{14} = \frac{A_{14}}{4032}; \quad \alpha_{15} = \frac{A_{15}}{1152}; \quad \alpha_{16} = \frac{A_{16}}{640},$$

and R_0, R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8 may be determined from the boundary conditions (17).

The bending moment M_ϕ in the first approximation may be calculated according to the following formula

$$M_\varphi = -D\left\{\nu_0 w_{rr}^{(0)} + \left(\frac{1}{r} w_r^{(0)} + \frac{1}{r^2} w_{\varphi\varphi}^{(0)}\right) + \lambda\left[\nu_0 w_{rr}^{(1)} + \right.\right.$$
$$\left.\left. + \left(\frac{1}{r} w_r^{(1)} + \frac{1}{r^2} w_{\varphi\varphi}^{(1)}\right)\right] - \lambda\left[s^{(0)}\left(\frac{w_r^{(0)}}{r} + \frac{w_{\varphi\varphi}^{(0)}}{r^2}\right) + t^{(0)} w_{rr}^{(0)}\right](1-\nu_0)\right\}.$$

As is known from linear theory (Ref. 5), the bending moment M_ϕ reaches the largest value on the hole profile at the points $\phi = \pm\frac{\pi}{2}$, and the concentration coefficient at these points is

$$\left(\frac{M_\varphi^{(0)}}{M}\right)_{r=a} = \frac{5+3\nu_0}{3+\nu_0}.$$

In our case, the expression for the bending moment M_ϕ on the profile at the points $\phi = \pm\frac{1}{2}\pi$ is

$$M_\varphi = M\frac{5+3\nu_0}{3+\nu_0}\left[1 - \frac{9M^2\lambda}{350(3+\nu_0)^3(5+3\nu_0)h^4}(8683\nu_0^4 + 72308\nu_0^3 + \right.$$
$$\left. + 206\,050\nu_0^2 + 222\,612\nu_0 + 45867)\right], \tag{18}$$

and the concentration coefficient at these points

$$\frac{M_\varphi}{M} = \frac{5+3\nu_0}{3+\nu_0}\left[1 - \frac{9M^2\lambda}{350(3+\nu_0)^3(5+3\nu_0)h^4}(8683\nu_0^4 + 72308\nu_0^3 + \right.$$
$$\left. + 206\,050\nu_0^2 + 222\,612\nu_0 + 45867)\right]$$

depends, as is known, on the external loading, the mechanical properties of the material, and the plate thickness.

Figure 2 presents a graph showing the change in the concentration coefficient of the moment M_ϕ over the inner profile at the points $\phi = \pm\frac{\pi}{2}$ as a function of the external moment for separate materials in the case h = 1 cm. /249
Curve 1 is for copper with the following characteristics

$$K = 1{,}33 \cdot 10^7\,\text{n/см}^2;\quad G = 0{,}47 \cdot 10^7\,\text{n/см}^2;\quad g_2 = 7{,}26 \cdot 10^6;$$
$$\lambda = 0{,}98 \cdot 10^{-7}\,\text{см}^4/\text{n}^2;$$

and curve 2 is for pure copper with the characteristics

$$K = 1{,}37 \cdot 10^7\,\text{n/см}^2;\quad G = 0{,}46 \cdot 10^7\,\text{n/см}^2;\quad g_2 = 0{,}18 \cdot 10^6;$$
$$\lambda = 0{,}255 \cdot 10^{-8}\,\text{см/n}^2;$$

Curve 3 is for open-hearth steel with the characteristics

$$K = 1{,}821 \cdot 10^7\,\text{n/см}^2.$$
$$G = 0{,}870 \cdot 10^7\,\text{n/см}^2;$$
$$g_2 = 0{,}085 \cdot 10^6;\ \lambda = 0{,}032 \cdot 10^{-8}\,\text{см}^4/\text{n}^2.$$

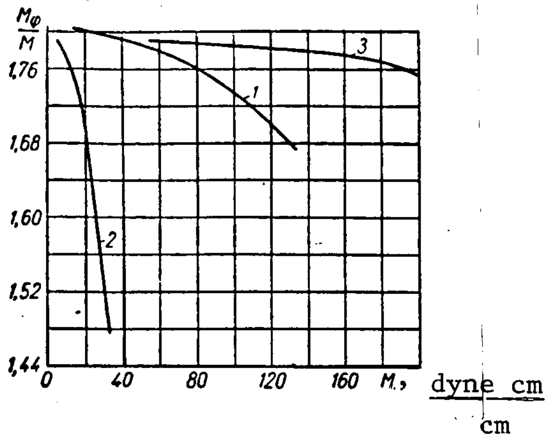

Figure 2

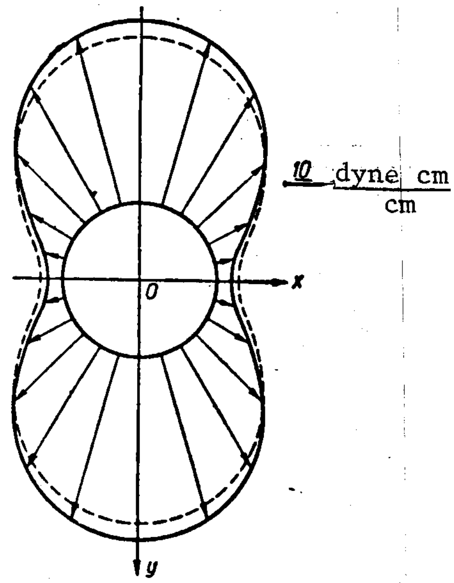

Figure 3

Figure 3 shows a diagram of the bending moment M_ϕ on the hole for a material with the following characteristics

$$K = 1{,}33 \cdot 10^7 \ n/cm^2; \ G = 0{,}47 \cdot 10^7 \ n/cm^2; \ \lambda = 0{,}98 \cdot 10^{-7} \ cm^4/n^2$$

and in the case M = 200 n·cm/cm, h = 1 cm, where the solid line designates the diagram of M_ϕ compiled on the basis of the linear theory, and the dashed line designates M_ϕ for our problem. The reduction in the concentration coefficient of the bending moment M_ϕ may be explained by the behavior of the material which is nonlinearly elastic. As may be seen from the diagram of the bending moment M_ϕ, there is an insignificant increase in the bending moment on the hole profile at the points $\phi = 0$, $\phi = \pi$. This is confirmed by a decrease in the bending moment M_ϕ at the dangerous locations due to the existence of sections which are loaded to a lesser extent.

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/ M5

STRESS CONCENTRATION NEAR CAVITIES IN AN INCOMPRESSIBLE MATERIAL /251

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General considerations. In previous articles (Ref. 1, 4), the formulation of the problem regarding plane deformations of an incompressible material has related the fundamental stresses (σ_1, σ_2) with the octahedral normal stresses (Σ), the octahedral shearing stresses (τ), and the stress functions (U) by the following formulas

$$\left.\begin{aligned} &\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = \Sigma \pm \sqrt{1,5}\,\tau;\\ &\tau^2 = 4p^2 U_{zz} U_{\bar z\bar z};\\ &\sqrt{\frac{2}{3}}\,\Sigma = 2pU_{z\bar z} + f;\\ &f = 2\sqrt{1,5}\int_0^{\vartheta_i} \tau(\vartheta_i)\,d\vartheta_i, \end{aligned}\right| \tag{1}$$

where p is the characteristic value of the stress; ϑ_1 -- deformation intensity.

We may select the law governing the change in the form as follows

$$\tau = G\,\mathrm{th}\,\vartheta, \quad \vartheta = 2\sqrt{1,5}\,\vartheta_i; \tag{2}$$

and we may represent the stress function by the series

$$U = {}^0U + {}^1U_\mu + {}^2U_\mu^2 + \dots \tag{3}$$

Let us confine ourselves to calculating the first two terms of the series. As was shown in (Ref. 4), if the region of the complex variable plane $z = x_1 + ix_2$, which is occupied by a body, is mapped onto the exterior of a unit circle of a plane by means of the following function

$$z = R\zeta + \sum_{k=0}^{\infty} c_k \zeta^{-k}, \tag{4}$$

then the problem may be reduced to the well known problem of successively /252
determining the biharmonic functions according to the boundary values on the region profile and at infinity.

The biharmonic function ${}^0U$ may be calculated according to the Goursat formula in terms of the two analytical functions (Ref. 2)

$${}^0U = \mathrm{Re}(\bar z\varphi_0 + \chi_0). \tag{5}$$

The function ${}^1U$ satisfies (Ref. 1) the following equation

$${}^1U_{zz\bar z\bar z} = -({}^0U_{zz}{}^0U_{\bar z\bar z})_{z\bar z} - \frac{1}{2}({}^0U_{zz\bar z}{}^0U_{z\bar z\bar z} - {}^0U_{zzz}{}^0U_{\bar z\bar z\bar z}). \tag{6}$$

Employing (5), we may find the particular solution (6):

$${}^1U^* = \frac{1}{8}[\varphi_0\bar\varphi_0 - (\bar z\zeta'\varphi_0' + \Psi_0)(z\bar\zeta'\bar\varphi_0' + \bar\Psi_0)], \tag{7}$$

where

$$\zeta' = \frac{d\zeta}{dz} = \frac{1}{z'(\zeta)}. \qquad (8)$$

We may express the biharmonic function ${}^1U$ - ${}^1U^*$ by means of the two new analytical functions

$${}^1U = \mathrm{Re}\,({}^1U^* + \bar{z}\varphi_1 + \varkappa_1). \qquad (9)$$

Let us now turn back to the formulation of the boundary conditions. The following must hold at the point on the profile with the outer normal (cos α, sin α) where there are no shearing or normal stresses (Ref. 2)

$$\frac{dU_{\bar{z}}}{ds} = -\frac{i}{2p} f e^{-i\alpha},$$

where ds is an element of arc length of the profile.

We thus obtain the following conditions

$$\left.\begin{array}{l} e^{-i\alpha}\dfrac{dU_{\bar{z}}}{ds} = -i\,{}^kF\mu^k; \\ {}^0F = 0;\ {}^1F = {}^0U_{zz}^2 e^{4i\alpha}. \end{array}\right\} \qquad (10)$$

Employing the boundary conditions

$$ds = -ie^{-i\alpha}\frac{d\sigma}{\sigma'};\quad \sigma = e^{i\theta}, \qquad (11)$$

let us formulate the boundary conditions for the analytical functions

$$\varphi_k(\sigma) + z(\sigma)\,\overline{\sigma'(\sigma)\,\varphi_k'(\sigma)} + \overline{\Psi_k(\sigma)} = -2\,{}^kU_{\bar{z}}^*(\sigma) - 2\int^k F(\sigma)\frac{d\sigma}{\sigma'(\sigma)} \qquad (12)$$
$$(k = 0,\ 1,\ \dots).$$

The calculation of the right hand parts of (12) may be simplified by the boundary conditions

$$\begin{array}{l} \Psi_0 = -\bar{\varphi}_0 - \bar{z}\sigma'\varphi_0'; \\ \Psi_0' = e^{-2i\alpha}(\sigma'\varphi_0' + \bar{\sigma}'\bar{\varphi}_0') - \bar{z}\sigma'(\sigma'\varphi_0')'. \end{array} \qquad (13)$$

We thus obtain /253

$$\begin{array}{l} \varphi_k(\sigma) + z(\sigma)\,\overline{\sigma'(\sigma)\,\varphi_k'(\sigma)} + \overline{\Psi_k(\sigma)} = g_k(\sigma);\ g_0 = 0; \\ g_1 = -\dfrac{1}{4}(\sigma'\varphi_0' + \bar{\sigma}'\bar{\varphi}_0')(\varphi_0 + \bar{\varphi}_0 e^{2i\alpha}) - \dfrac{1}{2}\displaystyle\int(\sigma'\varphi_0' + \bar{\sigma}'\bar{\varphi}_0')^2\frac{d\sigma}{\sigma'}. \end{array} \qquad (14)$$

Employing the well known methods presented in (Ref. 2, 3), we may write the analytical functions in the following form in terms of conditions at infinity

$$\begin{array}{l} \varphi_k = a_k\zeta + \varphi_k^*,\ \Psi_k = b_k\zeta + \Psi_k^*, \\ a_k = -R\,({}^kU_{\bar{z}\bar{z}}^{\infty} - {}^kU_{\bar{z}\bar{z}}^{*\infty}),\ b_k = -2R\,({}^kU_{zz}^{\infty} - {}^kU_{zz}^{*\infty}), \end{array} \qquad (15)$$

where ϕ_k^*, Ψ_k^* are functions which are holomorphic outside of a unit circle.

Since we are interested in stresses at the profile points, and not the stresses on the profile surfaces, we may calculate the second fundamental stress σ_t at the profile points according to the following formula

$$\sigma_t = 2\Sigma = p\,({}^0\sigma_t + {}^1\sigma_t\mu), \qquad (16)$$

where

$$ {}^0\sigma_t = 4\sqrt{1{,}5}\,{}^0U_{z\bar{z}}; $$
$$ {}^1\sigma_t = 4\sqrt{1{,}5}\,({}^1U_{z\bar{z}} + {}^0U_{zz}{}^0U_{\bar{z}\bar{z}}). $$

Employing (5), (9), (13), we obtain

$$ {}^0\sigma_t = 4\sqrt{1{,}5}\,\mathrm{Re}\,\sigma'\varphi_0'; \qquad {}^1\sigma_t = \sqrt{1{,}5}\,\mathrm{Re}\,[4\varphi_1' + \varphi_0\sigma'(\sigma'\varphi_0')'] + \frac{1}{2}\,{}^0\sigma_t^2. \tag{17} $$

Thus, it is sufficient to compute the boundary values ϕ_0 and ϕ_1 in order to determine the first two approximations of the concentration coefficients.

Let us investigate an infinitely extended body which has a cavity in the form of profile mapped onto the exterior of a unit arc by means of (4). The x_3-axis is directed along the cavity axis; therefore, the stress state in the x_1, x_2-plane is studied. The stress state at infinity is assumed to be given in the form of a compressed, uniformly distributed load having the intensity $2\sqrt{1.5}$ pq, forming the angle θ with the x_2-axis. We must determine the stress distribution around the surface of the cavity which is free of stress.

According to the fundamental stresses given at infinity

$$ \sigma_1^\infty = 0; \quad \sigma_2^\infty = -2\sqrt{1{,}5}\,pq \tag{18} $$

we may calculate the invariant quantities:

$$ \tau^\infty = pq; \quad \overset{\infty}{\Sigma} = -\sqrt{1{,}5}\,pq; \quad f^\infty = -\frac{1}{2\mu}\ln(1-\mu^2q^2). \tag{19} $$

We thus obtain the formulation of the conditions at infinity for the stress function /254

$$ 2\,{}^0U_{zz}^\infty = -qe^{-2i\theta}, \quad {}^1U_{zz}^\infty = 0, \ \ldots, \quad {}^0U_{z\bar{z}} = -\frac{1}{2}q $$
$$ {}^1U_{z\bar{z}}^\infty = -\frac{1}{4}q^2, \ \ldots \tag{20} $$

Employing (15) and (20), we obtain

$$ a_0 = \frac{qR}{2}; \quad b_0 = 2a_0\bar{\varepsilon}; \quad \varepsilon = e^{2i\theta}; \quad a_1 = \frac{a_0^2}{2R}; \quad b_1 = -\frac{a_0^2}{R}\bar{\varepsilon}. \tag{21} $$

The holomorphic portions of the function ϕ_k are determined below for particular cases.

Elliptical cavity. In this case, we have the following from (4) (Ref. 2)

$$ z = -R\left(\zeta + \frac{m}{\zeta}\right); \quad R = \frac{a+b}{2}; \quad m = \frac{a-b}{a+b}, \tag{22} $$

where a and b are the semimajor and semiminor axes of the ellipse.

The boundary conditions (14) may be reduced to the following form

$$ \varphi_k^*(\sigma) + \frac{1}{\sigma}\cdot\frac{m+\sigma^2}{1-m\sigma^2}\overline{\varphi_k'^*(\sigma)} + \overline{\Psi_k(\sigma)} = -a_k\left(\sigma + \frac{1}{\sigma}\cdot\frac{m+\sigma^2}{1-m\sigma^2}\right) + g_k. \tag{23} $$

Employing the method of Muskhelishvili, we obtain

$$\varphi_k^*(\zeta) = -a_k \frac{m}{\zeta} - \frac{b_k}{\zeta} + \frac{1}{2\pi i} \int_\gamma g_k \frac{d\sigma}{\sigma - \zeta}. \tag{24}$$

In the case k = 0 the classical solution is obtained (Ref. 2)

$$\varphi_0 = a\left(\zeta - \frac{m + 2\varepsilon}{\zeta}\right), \tag{25}$$

and the correcting component may be determined by the function

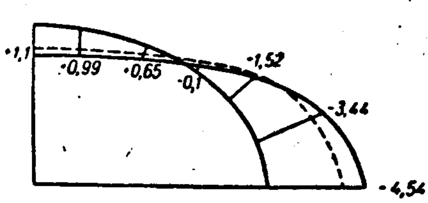
$$\varphi_1 = \frac{a_0^2}{2R}\left[\zeta + \left(\frac{2\varepsilon^2 + 2m\varepsilon}{m} + 2\varepsilon - m\right)\frac{1}{\zeta} - \frac{\varepsilon^2 + m\varepsilon}{m} \cdot \frac{\zeta}{\zeta^2 - m} + \right.$$
$$\left. + \frac{(\varepsilon + m)^2}{m\sqrt{m}} \ln \frac{\zeta - \sqrt{m}}{\zeta + \sqrt{m}}\right]. \tag{26}$$

Formula (16) is obtained for transverse stress at the profile points, where we must set

$$^0\sigma_t = -\frac{4a_0}{R} \cdot \frac{1 + 2m - m^2 \pm 2\cos 2t}{1 - 2m\cos 2t + m^2};$$
$$^1\sigma_t = \frac{4a_0^2}{R} \cdot [-2 \mp 16m - 8m^2 \pm 4m^3 + 6m^4 \mp 4m^6 +$$
$$+ (\mp 6 + 16m \mp 22m^2 \pm 10m^4 - 4m^5)\cos 2t +$$
$$+ (-4 \pm 4m - 12m^2 \mp 4m^3 + 4m^4 \cos 4t) + (4m \pm 2m^2)\cos 6t] \times$$
$$\times \left[(1 - 2m\cos 2t + m^2)^3 + \left(\frac{1 \mp 2m - m^2 \pm 2\cos 2t}{1 - 2m\cos 2t + m^2}\right)^2\right]^{-1}.$$

The superscript corresponds to $\theta = 0$; the subscript corresponds to $\theta = \frac{\pi}{2}$. /255

The figure presents the calculated stresses at the profile points of an elliptical cavity in the case m = 0.2 and $\mu = 0.5$. The dashed line designates the stresses calculated according to the classical formulas. The computational results point to a decrease in the classical concentration coefficient, amounting to 12%. In particular, in the case m = 0 the well known result (Ref. 4) is obtained for a circular cavity.



Cavity which is almost square. Developing the well known results of G. N. Savin (Ref. 3), we may set

$$z = -R\left(\zeta - \frac{1}{6} \cdot \frac{1}{\zeta^3}\right). \tag{27}$$

We may calculate ϕ_0, ϕ_1, and we obtain the following for the case $\theta = 0$

$$\varphi_0 = a_0\left(\zeta - \frac{12}{7} \cdot \frac{1}{\zeta} + \frac{1}{6} \cdot \frac{1}{\zeta^3}\right);$$
$$\varphi_1 = \frac{a_0^2}{2R}\left[\zeta + \frac{2}{7} \cdot \frac{1}{\zeta} + \frac{1}{6} \cdot \frac{1}{\zeta^3} - \frac{265}{147} \cdot \frac{\zeta}{1 + 2\zeta^4} + \right.$$
$$\left. + \int\left(\frac{46}{49} \cdot \frac{1}{1 + 2\zeta^4} + \frac{48}{7} \cdot \frac{\zeta^2}{1 + 2\zeta^4}\right) d\zeta\right]. \tag{28}$$

Formula (16) may be employed for the stress, $^0\sigma_t$ and $^1\sigma_t$ may be computed, and the stress values at individual points of the profile are as follows:

Angle, Degree	Stress	Angle, Degree	Stress
0	$-1{,}48-0{,}21\mu$	50	$-1{,}00+7{,}59\mu$
15	$-1{,}70-0{,}31\mu$	55	$0{,}27+4{,}97\mu$
30	$-2{,}70-1{,}94\mu$	60	$0{,}71+1{,}93\mu$
35	$-3{,}37-5{,}07\mu$	75	$0{,}84+0{,}46\mu$
40	$-3{,}86-8{,}95\mu$	90	$0{,}41+0{,}33\mu$
45	$-3{,}00-1{,}51\mu$		

Similar computations were performed for a triangular and rectangular cavity.

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BENDING OF REINFORCED PLATES

/256

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The articles (Ref. 1, 2, 5-9) investigated the problem of reinforcing plates with thin elastic rings, where the ring is regarded as a plate or as a solid fiber having the elastic characteristics of the reinforcing ring.

In the latter case, the actual profile of the junction is identified with the ring axis, which cannot be achieved many times in engineering technology.

Let us introduce the fundamental boundary relationships for the theory of reinforced plate bending, and let us discuss the above assumptions

Isotropic plate. Let the edge of a bounded, or unbounded, isotropic plate having the thickness h be reinforced by an isotropic ring having a variable transverse cross-section, one of whose main inertia axes lies in the middle plane of the plate. We shall employ the term transverse cross-section to designate the cross-section which is orthogonal to the profile of the junction L.

Assuming that the middle plane is the x0y-plane, we shall locate the origin at an arbitrary point on the plate, if it does have any holes, or in the middle of one of the holes if they do exist. Along L, we shall introduce a mobile coordinate system ($n\tau$) which is determined by the unit vector relative to $\vec{\tau}$ and by the unit vector of the normal $\vec{n}$ which is directed toward the plate exterior.

Without restricting the generality of the discussion, we shall assume that the ring is not influenced by the external stresses, and we shall regard it as an infinitely small element separated by two transverse cross-sections. Employing the generally accepted (Ref. 5) complex potentials $\phi(z)$, $\psi(z)$ and disregarding the ring axial deformation, we obtain the following boundary relationships from the condition of elastic equilibrium for the separated element and from the condition that the corresponding deformation components of the plate and the ring are equal along L, /257

$$(1-\nu)[\bar{z}\varphi''(z)+\psi'(z)]\dot{z}+\left[\left(1-\nu-\frac{b}{\rho}\right)\varphi'(z)-\right.$$
$$\left.-\left(3+\nu-\frac{b}{\rho}\right)\overline{\varphi'(z)}\right]\dot{\bar{z}}-ib[\varphi''(z)-\dot{\bar{z}}^2\overline{\varphi''(z)}]= \tag{1}$$
$$=\frac{i\dot{\bar{z}}}{\rho D}\left[\Pi_3(\theta)+\left(\rho-\frac{b}{2}\right)C_1\right];$$

$$[\bar{z}\varphi''(z)+\psi'(z)]\dot{z}+[\varphi'(z)+\overline{\varphi'(z)}]\dot{\bar{z}}=\frac{d}{ds}\left\{\dot{\bar{z}}\left[\gamma\left(\frac{b}{2\rho}-1\right)-\right.\right.$$
$$\left.\left.-\frac{\beta}{2}\left(\frac{d}{ds}-2i\right)-\frac{b}{2}\cdot\frac{d\beta}{ds}\right]\right\}, \tag{2}$$

where

$$\Pi_3(\theta)=\frac{d}{d\theta}(M_2+iH_2)-i(M_2+iH_2). \tag{3}$$

Here M_2 and H_2 are the bending moments and torque which are in operation in the ring transverse cross-section; b -- ring width; ρ -- radius of curvature of profile L; ν and D -- Poisson coefficient and plate cylindrical rigidity; $\dot{z} = \frac{\partial z}{\partial s}$, $i = \sqrt{-1}$; ds -- arc differential along L; θ -- angle at which the vector $\vec{n}$ intersects the 0_x-axis; β and γ -- angle of torsion and bending angle of the ring; and C_1 -- arbitrary real constant.

When deriving relationship (2), and from this point on, we assume that the differential of the ring axial line $ds = \left(\rho - \frac{b}{2}\right)d\theta$.

On the basis of the well known relationships of Clebsch-Kirchhoff (Ref. 4), the right hand sides of the boundary conditions (1), (2) are interrelated by the differential relationship

$$M_2 + iH_2 = \frac{C}{2\rho - b}\left\{(n+1)\left[\frac{d}{d\theta}(\gamma + i\beta) - i(\gamma + i\beta)\right] + \right. \\ \left. + (n-1)\left[\frac{d}{d\theta}(\gamma - i\beta) + i(\gamma - i\beta)\right]\right\}, \tag{4}$$

where the ring rigidity in bending is replaced by its rigidity in torsion by means of the relationship A = nC.

By way of an example, investigating the case of an infinite copper plate with a circular hole having the radius R reinforced by a steel ring having a constant transverse cross-section (height of the ring $h_1 = 1.5$, h =1.5 cm and R - b = 10 cm), we arrive at the conclusion that the solution to this problem which was obtained previously (Ref. 7, 8) may only be employed in the case $\alpha \leqq \frac{1}{20}$, where $\alpha = \frac{b}{2(R - b)}$.

Let us introduce into the investigation the concept of a *cylindrical stopper*, which will designate the particular case when the ring having a rectangular transverse cross-section degenerates into a rod in which $h_1 \leqq b \leqq$ /258
$\leqq 1.2h_1$, and $h \leqq h_1 \leqq 1.2h$.

Let us assume that the plate and the ring are made of one and the same material (copper). Assuming that $b = h_1$, $h_1 = h$, in the case of cylindrical bending by the moments $M_{x(\infty)} = M$ in the most dangerous cross-section $\left(\theta = \frac{\pi}{2}\right)$, we find that the following moments are in operation over the junction profile in the plate

$$M_\theta = 0{,}98M; \quad M_\rho = -0{,}08; \quad H_{\rho\theta} = 0. \tag{5}$$

It is clear from (5) that the plate under consideration functions like a solid plate. Consequently, the stopper model which we introduced takes into account the fundamental physical laws which are characteristic for problems of this type.

By way of an example, let us discuss the case when a steel stopper in which $b = h_1$ and $h_1 = 1.1h$ is soldered into the copper plate.

Cylindrical bending of an infinite plate. If the plate is bent by the moments $M_{x(\infty)} = M$, then the bending moments and the torque, which are in operation in the plate over the junction profile, have the values shown in Table 1.

TABLE 1

Ratio	θ, Degree							
	0	$\frac{\pi}{18}$	$\frac{\pi}{6}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$	$\frac{7\pi}{18}$	$\frac{8\pi}{18}$	$\frac{\pi}{2}$
$\frac{M_\theta}{M}$	0,345	0,349	0.381	0,430	0,453	0,472	0,484	0,489
$\frac{M_\rho}{M}$	1,600	1,539	1,091	0,406	0,074	—0,196	—0,373	—0,434
$\frac{M_{\rho\theta}}{M}$	—0,456	—0,428	—0,228	0,079	0,228	0,349	0,428	0,456

Twisting of an infinite plate. If a plate is twisted by the moment $H_{xy(\infty)} = H$, then the bending moments and the torque, which are in operation in the plate over the junction profile, have the values shown in Table 2.

Thus, as may be seen from these examples, the stopper represents a concentrator, and the moment M_ρ becomes a reference moment.

Aniostropic plate with an elliptical hole. Let us investigate an infinite anisotropic plate having the thickness 2h, at each point of which there is a plane of elastic symmetry which is parallel to the x0y-plane.

TABLE 2 /259

Ratio	θ, Degree							
	0	$\frac{\pi}{18}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{5\pi}{18}$	$\frac{7\pi}{18}$	$\frac{8\pi}{18}$	$\frac{\pi}{2}$
$\frac{M_\theta}{M}$	0	—0,049	—0,125	—0,144	—0,142	—0,093	—0,049	0
$\frac{M_\rho}{H}$	0	0,696	1,762	2,034	2,003	1,308	0,696	0
$\frac{H_{\rho\theta}}{H}$	0,911	0,856	0,455	0	—0,158	—0,698	—0,856	—0,911

Let us assume that the hole weakening the plate is reinforced by a thin elastic ring having a constant transverse cross-section and A = C. Without restricting the generality of the discussion, we shall assume that the ring is not influenced by external stresses, and the stress state of the plate is uniform at infinity.

As is known (Ref. 8), the boundary conditions of the formulated problem may be written in the following form

$$B_1\dot{z}_1\Phi'(z_1) + B_2\dot{z}_2\Psi'(z_2) + B_3\overline{\dot{z}_1\Phi'(z_1)} + B_4\overline{\dot{z}_2\Psi'(z_2)} + + R_1\Phi(z_1) + R_2\Psi(z_2) + R_3\overline{\Phi(z_1)} + R_4\overline{\Psi(z_2)} = 0, \tag{6}$$

where $z_j = x + \mu_j y$, $\dot{z}_j = \frac{dz_j}{ds}$, μ_j -- complex parameters of the plate (j = 1, 2); the constants B_k, R_k (k = 1, 2, 3, 4) may be expressed by means of the elastic constants of the plate (Ref. 8), and the functions $\Phi(z_1)$ and $\Psi(z_2)$ are as follows (Ref. 5):

$$\Phi(z_1) = H_0 z_1 + \Phi_0(z_1); \quad \Psi(z_2) = (H_1 + iH_3) z_2 + \Psi_0(z_2), \tag{7}$$

where H_0, H_1, H_3 are the given constants and $\Phi_0(z_1)$, $\Psi_0(z_2)$ have the expansions

$$\Phi_0(z_1) = \sum_1^\infty b_k z_1^{-k}; \quad \Psi_0(z_2) = \sum_1^\infty b_k^1 z_2^{-k}. \tag{8}$$

Directing the Ox-axis along the semimajor axis of the ellipse, employing the relationship

$$z = \omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right), \quad 0 \leqslant m < 1 \tag{9}$$

we may map the plate region onto the exterior of a unit circle γ, in which $\zeta = \sigma$.

On the basis of (9), we have the following for points on the junction profile

$$z_j = \frac{R}{2}\left[(1 - i\mu_j)\left(\sigma + \frac{m}{\sigma}\right) + (1 + i\mu_j)\left(m\sigma + \frac{1}{\sigma}\right)\right]; \tag{10}$$

and, taking (8), (10) into account, we may rewrite (6) in the following form /260

$$i\sigma[B_1\varphi_0'(\sigma) + B_2\psi_0'(\sigma)] - \frac{i}{\sigma}[B_3\overline{\varphi_0'(\sigma)} + B_4\overline{\psi_0'(\sigma)}] + |\omega'(\sigma)|[R_1\varphi_0(\sigma) + + R_2\psi_0(\sigma) + R_3\overline{\varphi_0(\sigma)} + R_4\overline{\psi_0(\sigma)}] = \frac{i}{\sigma}[\bar{K}_1 B_3 + \bar{K}_2 B_4] - - i\sigma[K_1 B_1 + K_2 B_2] - |\omega'(\sigma)|\left[(K_1 R_1 + K_2 R_2)\sigma + + \frac{1}{\sigma}(\bar{K}_1 R_3 + \bar{K}_2 R_4)\right], \tag{11}$$

where

$$K_1 = \frac{RH_0}{2}[1 - i\mu_1 + m(1 + i\mu_1)], \quad K_2 = \frac{R(H_1 + iH_3)}{2} \times \times [1 - i\mu_2 + m(1 + i\mu_2)], \tag{12}$$

and the functions $\phi_0(\sigma) = \sum_1^\infty a_k \sigma^{-k}$, $\psi_0(\sigma) = \sum_1^\infty a_k^1 \sigma^{-k}$ satisfy the condition

$$\Phi(z_1) = K_1\sigma + \varphi_0(\sigma); \quad \Psi(z_2) = K_2\sigma + \psi_0(\sigma). \tag{13}$$

Assuming that $|\omega'| = R\,[1 - \frac{m}{2}(\sigma^2 + \sigma^{-2})]$, we obtain the following by the method of N. I. Muskhelishvili from (11) and the equation connected with it

$$[\lambda(\zeta^2+\zeta^{-2})-R][R_1\varphi_0(\zeta)+R_2\psi_0(\zeta)]-i[B_1\varphi_0'(\zeta)+B_2\psi_0'(\zeta)]\zeta-$$
$$-\lambda[R_1a_2+R_2a_2^1+(R_1a_1+R_2a_1^1)\zeta+(R_3\bar{a}_1+R_4\bar{a}_1^1)\zeta^{-1}]=$$
$$=[R(\overline{K}_1R_3+\overline{K}_2R_4)-i(\overline{K}_1B_3+\overline{K}_2B_4)-\lambda(K_1R_1+K_2R_2)]\zeta^{-1}-$$
$$-\lambda[\overline{K}_1R_3-\overline{K}_2R_4]\zeta^{-3}; \tag{14}$$
$$[\lambda(\zeta^2+\zeta^{-2})-R][\overline{R}_3\varphi_0(\zeta)+\overline{R}_4\psi_0(\zeta)]-i[\overline{B}_3\varphi_0'(\zeta)+\overline{B}_4\psi_0'(\zeta)]\zeta-$$
$$-\lambda[\overline{R}_3a_2+\overline{R}_4a_2^1+(\overline{R}_3a_1+\overline{R}_4a_1^1)\zeta-(\overline{R}_1\bar{a}_1+\overline{R}_2\bar{a}_1^1)\zeta^{-1}]=$$
$$=[R(\overline{K}_1\overline{R}_1+\overline{K}_2\overline{R}_2)-i(\overline{K}_1\overline{B}_1+\overline{K}_2\overline{B}_2)-\lambda(K_1\overline{R}_3+K_2\overline{R}_4)\zeta^{-1}-$$
$$-\lambda[\overline{K}_1\overline{R}_1+\overline{K}_2\overline{R}_2]\zeta^{-3}. \tag{15}$$

By comparing the coefficients for identical powers of the variables in (14), (15) we obtain an infinite system, from which we may readily find all a_{2n+1}, a^1_{2n+1} ($n = 1, 2, \ldots$), employing the well known constants a_1, a^1_1, $\bar{a}_1$, $\bar{a}^1_1$. In determining these quantities, we may make use of the fact that the functions $\psi_0(\zeta)$ and $\phi_0(\zeta)$ are analytical outside of γ, i.e.,

$$\lim a_k=0;\quad \lim a_k^1=0. \tag{16}$$

Since, for the given degree of computational accuracy, the index ℓ always exists, which makes it possible to set the following, without disturbing the accuracy,

$$a_i=0,\quad a_i^1=0\quad (i=2n+1>l), \tag{17}$$

combining the adjoint equations with (17), we may thus determine the coefficients a_1, a^1_1, $\bar{a}_1$, $\bar{a}^1_1$. /261

Similarily, the coefficients a_{2n+1}, a^1_{2n+2} ($n = 1, 2, \ldots$) may be expressed by means of a_2, a^1_2, $\bar{a}_2$, $\bar{a}^1_2$; in order to determine these quantities, we must set $m = 2n + 2$ in condition (17).

The functions $\Phi(z_1)$, $\Psi(z_2)$ have now been found with the determination of $\phi_0(\zeta)$, $\psi_0(\zeta)$ according to (13), and the solution of the problem under consideration is concluded.

For the particular case of the problem when $m = 0$, we obtain (Ref. 9)

$$\varphi_0(\zeta)=a_1\zeta^{-1};\quad \psi_0(\zeta)=a_1^1\zeta^{-1}, \tag{18}$$

where

$$a_1=\frac{1}{\Delta}\{[(RR_3-iB_3)\overline{K}_1+\overline{K}_2(RR_4-iB_4)](i\overline{B}_4-R\overline{R}_4)-$$
$$-(iB_2-RR_2)[(R\overline{R}_1-i\overline{B}_1)\overline{K}_1+\overline{K}_2(R\overline{R}_2-i\overline{B}_2)]\}; \tag{19}$$
$$a_1^1=\frac{1}{\Delta}\{[(R\overline{R}_1-i\overline{B}_1)\overline{K}_1+\overline{K}_2(R\overline{R}_2-i\overline{B}_2)](iB_1-RR_1)-$$
$$-i(\overline{B}_3-R\overline{R}_3)[(RR_3-iB_3)\overline{K}_1+\overline{K}_2(RR_4-iB_4)]\}; \tag{20}$$
$$\Delta=[(iB_1-RR_1)(i\overline{B}_4-R\overline{R}_4)-(iB_2-RR_2)(i\overline{B}_3-R\overline{R}_3)]\neq 0. \tag{21}$$

Table 3 presents the values of the moments M_x, M_y, H_{xy} for high quality veneer, and the Ox-axis is directed along the core fibers, m = 0; $M_{x(\infty)} = M$; $M_{y(\infty)} = H_{xy(\infty)} = 0$, and the reinforcing ring is copper with a height of $h_1 = 2.2h$ and a width of $b = 1.86h_1$. The angle θ is read off from the Ox-axis counterclockwise.

TABLE 3

Ratio	θ, Degree							
	0	$\frac{\pi}{18}$	$\frac{\pi}{6}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$	$\frac{7\pi}{18}$	$\frac{8\pi}{18}$	$\frac{\pi}{2}$
$\frac{M_x}{M}$	1,085	1,155	1,374	1,583	1,353	1,298	1,162	0,991
$\frac{M_y}{M}$	0,034	0,018	—0,125	—0,043	—0,057	—0,070	—0,084	—0,091
$\frac{H_{xy}}{M}$	—0,091	—0,088	—0,173	—0,107	0,010	0,048	0,090	0,134

Comparing the data given in Table 3 with the computational results for the same plate, but with a non-reinforced hole (Ref. 3), we find that the ring reduces the stress conccentration coefficient in the zone of the hole by a factor greater than two.

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A METHOD OF DETERMINING THE STRESS CONCENTRATION AT "NODAL POINTS" /263

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Very little research has been devoted to the problem of the stress field perturbation around holes or external chamfers. The most detailed examination has been made of the plane problem of elasticity theory concerning stress concentration in an elastic medium around the exterior of a curve L (Ref. 5). It may be assumed that the solution of this problem does not entail any particular difficulties, since sufficiently effective methods have been developed (Ref. 6, 10) to formulate the function mapping a circle onto the exterior of the given curve L. Much greater difficulties arise when dealing with the problem of stress concentration around holes in an infinite region reinforced by elastic rings, the problem of stress concentration in the case of doubly-connected regions, the problem with mixed boundary conditions, the elasto-plastic problem, etc.

The Structural Mechanics Department of Gor'kiy Institute of Structural Engineers has recently devoted a great deal of research to stress concentration in elements of engineering constructions, and has solved many numerical examples [see the summary in (Ref. 7), and also see Trudy Gor'kovskogo Inzhenerno-Stroitel'noy Instituta im. V.P. Chkalova, No. 39, 1961 and No. 44, 1963, 1964].

Other research has been devoted to such problems as the following:

1. Torsion and bending of box-like welded and curved profiles.

2. Torsion and bending of orthotropic rods with doubly-connected transverse cross-section.

3. Stress concentration around holes and external chamfers in compressed compounds.

4. Stress concentration around small holes of the observational type in semi-infinite blocks and around chamfers on the outer boundary.

5. Stress concentration in the teeth of toothed wheels. /264

The method of electromodeling of conformal mappings (Ref. 8, 9) was employed in these studies to formulate the function $z = \omega(\zeta)$, conformal mapping the circle $|\zeta| < 1$ [circular ring] onto the given simply-connected [doubly-connected] region. Electronic computers were employed to overcome the difficulties entailed in the calculations.

We have significantly changed the method of formulating the conformal mapping functions (Ref. 6). Interpolation Lagrange polynomials in the complex region lie at the basis of the analytical section, instead of trigonometric polynomials which approximate only the real portion of the boundary value of the auxiliary function $u + iv = \frac{z}{\zeta}$.

The initial studies performed by I. I. Serebryakov have shown that the formulation of mapping functions, by means of electromodeling and Lagrange interpolation polynomials, has signficant advantages in solving the problems of elasticity theory. At the "intermediate" points the deviations of the given boundary L and the boundary L', corresponding to the formulated function $z = \omega_n(\zeta)$, are thus smaller and -- which is the essential point -- "oscillations" of the stress curves, caused by local deviations in the radii of the boundary curvatures L and L', practically disappear.

We cannot deal with this problem in greater detail here, since the purpose of this article is to investigate another problem.

The development of research on the plane problem of elasticity theory shows that methods based on employing the complex variable functions and the method of N. I. Muskhelishvili (Ref. 3) are most effective in studying the stress concentration. However, these methods cannot be successfully applied in every case to solve the applied problems. In particular, great difficulties are encountered in solving mixed problems for doubly-connected regions with composite boundaries, etc.

The method of finite differences (and several of its modifications) has been extensively used in recent years. This method makes it possible to determine satisfactorily the stress state, but it leaves the problem open regarding stress concentration at nodal points with a given radius of curvature-around outer chamfers and inner junctions at the hole boundary.

We shall assume that the method of finite differences (Ref. 1) or p-transformations (Ref. 4) may be employed to solve the plane problem for an elastic medium occupying a certain region S in the plane $z = x + iy$. A portion of the boundary of this region -- curve L -- has a "nodal" point with a given transitional form b from section a to section c (Figure 1). Curve L may be /265
both the boundary of the hole, and the outer boundary of the region. We shall also assume that the boundary conditions are uniform on the section b and on the ends of the sections a and c which are adjacent to it. We shall divide a portion of the region S -- region S_1 in the vicinity of the "nodal" point -- of the section b by the curve L. The boundary L_1 of the simply-connected region will consist of a section of the boundary L -- section b and the adjacent ends of sections a and c -- and the boundary L_2 lying within the region S.

The boundary conditions at L are known from the formulation of the problem, and the boundary conditions at L_2 may be approximately determined by the method of finite differences. If we select the boundary L_2 sufficiently far from the "nodal" point -- section b -- then the magnitude of the radius of curvature will not influence the boundary conditions at L_2.

As a result, we obtain the first (or second) fundamental problem: to determine the stress state in an elastic medium occupying the simply-connected region S in the plane z with boundary L_1, at which the external stresses are given (or displacements).

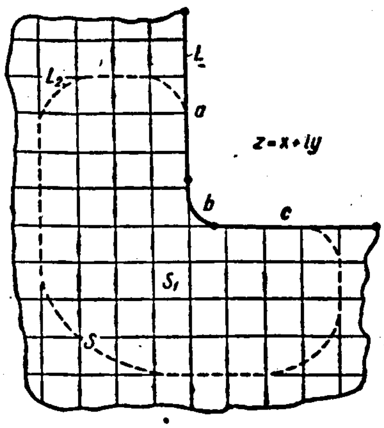

Figure 1

As is well known, the solution of this problem may be reduced to determining the function $z = \omega(\zeta)$ conformally mapping the circle $|\zeta| < 1$ onto the region S_1, and then to finding the functions $\phi(\zeta)$ and $\psi(\zeta)$ which are regular in the region of a unit circle and which satisfy the following condition at $|\zeta| = 1$

$$\eta\varphi(\sigma) + \frac{\omega(\sigma)}{\bar{\omega}'(\bar{\sigma})}\bar{\varphi}'(\bar{\sigma}) + \bar{\psi}(\bar{\sigma}) = f(\sigma) + C. \quad (1)$$

Here $\sigma = e^{i\theta}$ is the boundary value of $\zeta = \rho e^{i\theta}$; $f(\sigma)$ -- the function given on γ.

For the first fundamental problem, we have

$$f(\sigma) = i\int_0^s (X_n + iY_n)\,ds; \quad (2)$$

and for the second fundamental problem we have

$$f(\sigma) = -2\mu(u_0 + iv_0), \quad (3)$$

where X_n, Y_n are the components of the external (at L_1) stresses; u_0, v_0 -- the displacement components on L_1; $\eta = 1$ for the first fundamental problem; $\eta = -\varkappa$ for the second fundamental problem, where $\varkappa$ is the elastic constant introduced by G. V. Kolosov. /266

In the case when the mapping function $z = \omega(\zeta)$ has the form of a polynomial $z = \omega_n(\zeta)$, the complex potentials $\phi(\zeta)$ and $\psi(\zeta)$ may be determined, as is known, in closed form by the method of N. I. Muskhelishvili (Ref. 3).

In the case under consideration, the solution will have certain singularities. In the first place, the form and location of the ends of the section boundary L_2 can be selected arbitrarily, which considerably facilitates the formulation of the mapping function $z\omega_n(\zeta)$, and sometimes the calculation of the functions (2), (3).

In the second place, the functions $f(\sigma)$ cannot be determined analytically from the solution in finite differences: only the values of the function (3) are known at the nodes of the grid, if the displacement problem is solved, or the values σ_x, σ_y, τ_{xy} at the nodes of the grid, if the stress problem is solved. Therefore, both in the case of the first fundamental problem and in the case of the second fundamental problem, it is necessary to formulate the interpolation polynomial in a complex form $f_n(\zeta)$, which would coincide with the value of the function $f(\sigma)$ at the given points (interpolation nodes).

If we employ the points M_j of the boundary L_1, corresponding to $\zeta = \zeta_j = e^{i\theta_j}$, where $\theta_j = \frac{2\pi}{m}j$ ($j = 1, 2, \ldots, m$), as the interpolation nodes, then the coefficients of the interpolation polynomial

$$f_n(\sigma) = \sum_{k=-\frac{m}{2}}^{\frac{m}{2}-1} A_k \sigma^k \tag{4}$$

may be determined from the expression

$$A_k = \frac{1}{m} \sum_{j=1}^{m} f_j e^{-ik\theta_j}, \tag{5}$$

where $f_j = f(\theta_j)$ is the value of the function $f(\sigma)$ at the interpolation nodes $\zeta = \zeta_j$.

If the initial problem is solved for displacements, then the value of the function (3) at the points

$$z_j = \omega_n(e^{i\theta_j}) \quad (j = 1, \ldots, m) \tag{6}$$

is obtained by simple interpolation of the known values of the function (3) at the nodes of the grid.

If the initial problem is solved for stresses, in order to determine the function (2) at the interpolation nodes, it is necessary to replace the integral (2) by the finite sum /267

$$f_j = \sum_{\nu=1}^{\nu=j} \left[\frac{\sigma_y^{(\nu)} + \sigma_y^{(\nu-1)}}{2} (x_\nu - x_{\nu-1}) - \frac{\tau_{xy}^{(\nu)} - \tau_{xy}^{(\nu-1)}}{2} (y_\nu - y_{\nu-1}) \right] + i \sum_{\nu=1}^{\nu=j} \left[\frac{\sigma_x^{(\nu)} + \sigma_x^{(\nu-1)}}{2} (y_\nu - y_{\nu-1}) - \frac{\tau_{xy}^{(\nu)} - \tau_{xy}^{(\nu-1)}}{2} (x_\nu - x_{\nu-1}) \right], \tag{7}$$

where $\sigma_x^{(\nu)}$, $\sigma_y^{(\nu)}$, $\tau_{xy}^{(\nu)}$ are the stress values at the points $z_\nu = z_j = x_\nu + iy_\nu$, determined by interpolation over the values of these quantities at the grid nodes.

We should note that an increase in functions (2) or (7) when passing around the contour L_1 would have to equal zero (principal vector equals zero), and the coefficients A_k of the function (4) which are found must satisfy the following equation (principal moment equals zero)

$$\operatorname{Im} \sum_{k=1}^{p} k \bar{C}_k A_k = 0,$$

where C_k ($k = 1, 2, \ldots, n$) are the coefficients of the polynomial $z = \omega_n(\zeta)$, /268
p -- larger of the quantities $\frac{m}{2} - 1$ or n.

By way of an example, let us study the stress concentration around a

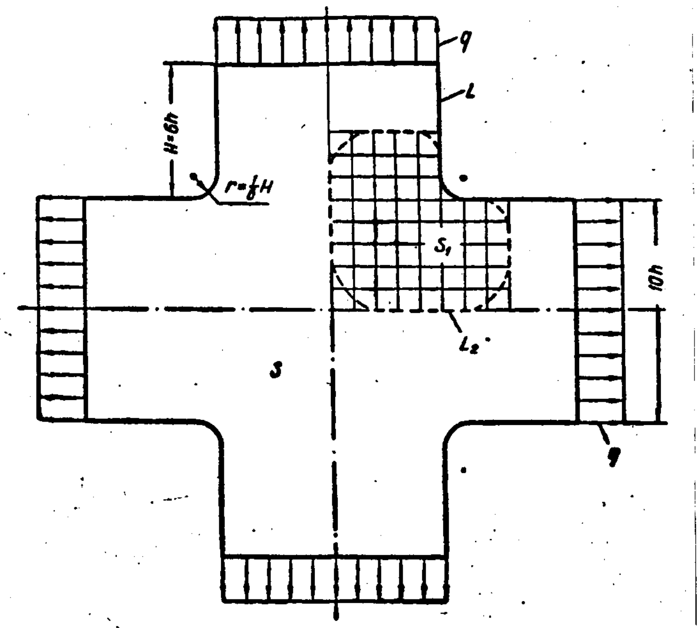

Figure 2

"right angle", when the radius of curvature $r = \frac{1}{6} H = h$. The solution of the problem in finite differences was taken from the book by L. I. Dyatlovitskiy (Ref. 2, Figure 61, 62).

The boundary L_1 of the region S is plotted in Figure 2. Figure 3 shows a graph of the stress σ_θ at points of the radial transition boundary.

In particular, at the dangerous point max σ_θ = 2.403q. For purposes of comparison, we shall write the values of max σ_θ presented in the monograph (Ref. 2): max σ_θ = 2.204q -- for a large grid employing the formula of Brass; max σ_θ = 2.220q -- for a grid with a non uniform step.

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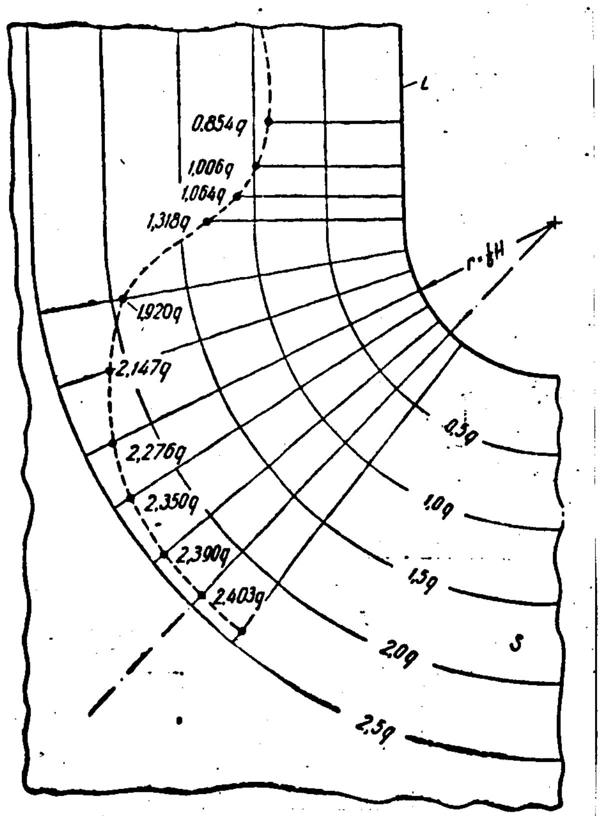

Figure 3

(Numerical Solution of Two-Dimensional and Three-Dimensional Boundary Value Problems of the Mathematical Function and the Discrete Argument Function). Izdatel'stvo Kiyevskogo Gosudarstvennogo Universiteta, 1962.

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SOLUTION OF SEVERAL PROBLEMS FOR DOUBLY CONNECTED REGIONS WITH CONTIGUOUS CIRCULAR BOUNDARIES /270

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This report presents the results derived from solving certain problems for doubly-connected regions with circular boundaries. The solutions presented in the literature for the problems investigated below are not valid when the boundaries of the region are located close to each other. A method given in (Ref. 2) is used to formulate valid solutions.

Extension of an eccentric ring by two concentrated forces. Let us assume that we must determine the stress state of an eccentric ring pulled along the line of symmetry by two concentrated forces applied from the outer profile. The inner ring radius will be designated by r_1, the outer ring radius -- by r_2, the thickness of the connector -- by h, and we shall assume that the intensity of each force equals P (see the figure).

The study (Ref. 6) investigated this problem by means of bipolar coordinates and the Fourier method. However, the result obtained in this study is not valid, when the ratio $\eta = h/r_1$ is small.

In order to formulate a valid solution, we shall employ the method advanced in (Ref. 2). For this purpose, we shall relate the region S, occupied by the ring, to the bipolar coordinate system α, β (Ref. 3). We shall assume that $\alpha = \alpha_1$ on the inner profile, and $\alpha = \alpha_2$ on the outer profile.

Employing the general solution (Ref. 3) of G. Jeffery, we obtain

$$g\Phi = \Phi_0 + \Phi_1, \tag{1}$$

where Φ is the Airy stress function;

$$\Phi_0 = \frac{P \operatorname{ch} \alpha_2}{2\pi \operatorname{sh} \varepsilon (\operatorname{sh}^2\alpha_1 + \operatorname{sh}^2\alpha_2)} \{2\alpha \operatorname{ch} \varepsilon (\operatorname{ch} \alpha - \cos\beta) + \\ + \cos\beta [\operatorname{sh} \varepsilon (2\lambda - 1) + \operatorname{ch} \varepsilon \operatorname{sh} 2\alpha_1 - \operatorname{sh} \varepsilon]\}; \tag{2}$$

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$$\Phi_1 = -\frac{P \operatorname{ch} \alpha_2}{\pi} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{f(x_k) e^{2ik\beta}}{(x_k^2 - \varepsilon^2) \Delta(x_k)}; \tag{3}$$

$$f(x) = (x^2 - \varepsilon^2) \operatorname{sh} \varepsilon \operatorname{sh} \lambda\varepsilon \operatorname{sh} x \operatorname{sh} \lambda x + \varepsilon^2 (mx \operatorname{ch} x + \\ + \operatorname{ch} \varepsilon \operatorname{sh} x) [\operatorname{ch} \lambda\varepsilon \operatorname{sh} \lambda x - (x/\varepsilon) \operatorname{sh} \lambda\varepsilon \operatorname{ch} \lambda x], \\ \Delta(x) = \operatorname{sh}^2 x - m^2 x^2; \tag{4}$$

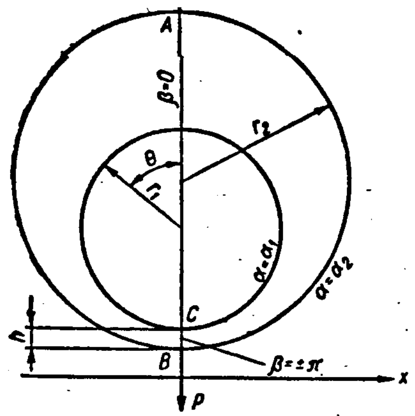
$$\varepsilon = \alpha_1 - \alpha_2, \quad \alpha_1 - \alpha = \lambda\varepsilon, \quad x_k = 2\varepsilon k; \tag{5}$$

$$g = a^{-1} (\operatorname{ch} \alpha - \cos\beta) \quad (0 \leqslant \lambda < 1, \ -\pi \leqslant \beta \leqslant \pi). \tag{6}$$

We may determine the parameters α_1, α_2, a by the following relationships

$$\operatorname{sh}^2 \frac{\alpha_1}{2} = \frac{\eta\gamma}{2(\gamma - 1)}, \tag{7}$$

$$\operatorname{sh}^2\frac{\alpha_2}{2} = \frac{\eta}{2\gamma(\gamma-1)} \; (\gamma = r_2/r_1, \; \eta = h/r_1), \quad \operatorname{sh}\alpha_1 = a/r_1, \quad \operatorname{sh}\alpha_2 = a/r_2. \tag{7}$$



Solution (3) is not valid for two reasons. In the first place, due to the singularities on the outer profile, the series (3) converges nonuniformly in the ring region, and it even diverges at the points where the forces are applied. This may be readily eliminated (Ref. 6) by isolating the solution for the disk without a hole, whose radius coincides with the outer circle radius, and which is subjected to the influence of the same concentrated forces. In the second place, the convergence of the series depends essentially on the magnitude of the parameter ε which in its turn depends on η, in particular. Therefore, even a solution which is transformed by this method is poorly suited for calculations in the case of small ε.

Let us now transform the series (3) according to the formulas given in (Ref. 2), after which we obtain

$$\Phi_1 = \Phi_1^{(0)} + \Phi_1^{(1)} + \Phi_1^{(2)}. \tag{8}$$

Here

$$\Phi_1^{(0)} = -\frac{P}{\pi^2(m^2-1)}\left[\lambda \operatorname{sh}\varepsilon \operatorname{sh}\lambda\varepsilon - (m + \operatorname{ch}\varepsilon)\left(\lambda \operatorname{ch}\lambda\varepsilon - \frac{\operatorname{sh}\lambda\varepsilon}{\varepsilon}\right)\right]; \tag{9}$$

$$\Phi_1^{(1)} = -\frac{P}{\pi\varepsilon}\int_{-\infty}^{\infty}\frac{f(x)\,e^{i\varphi x}}{(x^2-\varepsilon^2)\,\Delta(x)}\,dx \quad (\varphi = \beta/\varepsilon); \tag{10}$$

$$\Phi_1^{(2)} = \frac{8P}{\varepsilon}\,\operatorname{Im}\sum_k \frac{f(z_k)\cos z_k\varphi \exp(i\pi z_k/\varepsilon)}{\Delta'(z_k)\,[1-\exp(i\pi z_k/\varepsilon)]}. \tag{11}$$

In formula (11), z_k are complex zeros of $\Delta(z)$ lying in the first fourth /272
of the plane $z = x + iy$ (Ref. 1).

The solution (8) - (11) is valid for the interval $\beta \in (-\pi, \pi)$. In order to have a solution in the vicinity of $\beta = \pi$, we must replace β by $\beta' = \beta - \pi$ in formulas (10), (11). This substitution is permissible, since the function Φ_1 has a period equalling π, as follows from (3).

Determining the stress field by means of formulas (8) - (11), we should note that the field is symmetrical with respect to the cross-sections $\beta = 0$ and $\beta = \pi$. In view of this fact, when studying this field, it is sufficient for us to limit ourselves to the segments $\beta, \beta' \in [0, \pi/2]$. Therefore, if we take the fact into account that $\operatorname{Im} z_k \geqq 0$ 4.212-0.0338ε^2 ($k = 1, 2, \ldots$), in the case of $\varepsilon \leq 1$ we may disregard the function $\Phi_1^{(2)}$ and the stresses corresponding to it. As may be seen, the validity of relationships (9), (10) does not depend on ε.

We may obtain comparatively simple formulas for a numerical analysis of stress. By way of an example, we shall present the computational formulas for σ_β on the inner ring circle.

For $0 \leq \beta \leq \pi/2$, we have

$$\sigma_\beta = \frac{P(\operatorname{ch}\alpha_1 - \cos\beta)}{\pi r_1 \operatorname{sh}\alpha_1}\left[\frac{2\operatorname{ch}\alpha_2}{\operatorname{sh}\varepsilon\,(\operatorname{sh}^2\alpha_1 + \operatorname{sh}^2\alpha_2)}(\operatorname{ch}\varepsilon\operatorname{sh}\alpha_1 + \operatorname{sh}\varepsilon\cos\beta) - \frac{2m}{m^2-1} + \Gamma\right]. \tag{12}$$

In the case $0 \leq \phi \leq 0.4$, we have

$$\Gamma = (4m/\varepsilon)\,[0{,}411\,(1 - 0{,}210\varepsilon^2) + 8{,}599\varphi^2\,(1 - 0{,}0913\varepsilon^2) - 10{,}41\varphi^4\,(1 + 0{,}0291\varepsilon^2) + 14{,}08\varphi^6], \tag{13}$$

and in the case $0.4 \leq \phi \leq \pi/2\varepsilon$

$$\begin{aligned}\Gamma = \frac{4\pi m}{\varepsilon}\Big\{&\frac{\varepsilon\sin\beta}{2(\operatorname{ch}\varepsilon - m)} - e^{-b_1\varphi}\,[(1{,}086 + 0{,}0103\varepsilon^2)\sin a_1\varphi - \\ &-(0{,}203 + 0{,}0847\varepsilon^2)\cos a_1\varphi] + e^{-b_2\varphi}\,[(1{,}037 + 0{,}0054\varepsilon^2)\sin a_2\varphi - \\ &-(0{,}1225 + 0{,}0455\varepsilon^2)\cos a_2\varphi] - e^{-b_3\varphi}\,[(1{,}022 + 0{,}0034\varepsilon^2)\sin a_3\varphi - \\ &-(0{,}0883 + 0{,}0312\varepsilon^2)\cos a_3\varphi]\Big\}; \quad a_1 = 2{,}251 + 0{,}1811\varepsilon^2; \\ a_2 = 2{,}769 + 0{,}1729\varepsilon^2;\ & a_3 = 3{,}103 + 0{,}1703\varepsilon^2;\ b_1 = 4{,}212 - 0{,}03385\varepsilon^2; \\ b_2 = 7{,}498 - 0{,}02042\varepsilon^2;\ & b_3 = 10{,}712 - 0{,}01472\varepsilon^2.\end{aligned} \tag{14}$$

The formulas for σ_β in the case $0 \leq \beta' \leq \pi/2$ may be obtained by replacing ϕ by $\phi' = \beta'/\varepsilon$ in (13), (14).

Calculations based on the formulas obtained point to the strong stress concentration in the vicinity of the connector. Thus, for a ring with the parameters $\gamma = 2$ and $\eta = 0.09$ $\dfrac{\sigma_\beta(\alpha_1, \pi)}{\sigma_\beta(\alpha_1, 0)} = 14.4$.

Let us now study the behavior of σ_β in the case $h \to 0$ (we shall assume /273
that r_1 and r_2 are specified). The following relationships follow from (7)

$$\alpha_1 = k_1\sqrt{\eta} + O(\eta); \quad \alpha_2 = k_2\sqrt{\eta} + O(\eta) \quad \left(k_1 = \sqrt{\frac{2\gamma}{\gamma - 1}},\ k_2 = \sqrt{\frac{2}{\gamma(\gamma - 1)}}\right). \tag{15}$$

In addition, if we introduce the polar angle θ, just as in the figure, we find that in the case $\theta \in [0, \pi - \delta]$ the following asymptotic equations hold

$$\beta = k_1\sqrt{\eta}\operatorname{tg}\theta/2 + O(\eta^{3/2}), \quad \operatorname{ch}\alpha_1 - \cos\beta = \frac{k_1^2\eta}{1 + \cos\theta} + O(\eta^{3/2}). \tag{16}$$

In the case $h \to 0$, the constant δ may be chosen arbitrarily small. We should note that $\beta = \pm\,\pi$ is independent of h in the case $\theta = \pm\,\pi$.

By means of relationships (12) - (16) we may readily establish the form of σ_β^* -- of the principal term σ_β. For $0 \leq \theta \leq \pi - \delta$, we have

$$\sigma_\beta^* = \frac{P\Delta_0\eta^{-1/2}}{\pi r_1(1+\cos\theta)}; \quad \Delta_0 = \frac{2k_1(2k_1-k_2)}{k_1^2+k_2^2} - \frac{6k_1}{(k_1-k_2)^2}. \tag{17}$$

The latter relationship shows that in the case $\eta \to 0$ $\sigma_\beta \to \infty$, having the order $\eta^{-1/2}$ on the given segment.

In the vicinity of $\theta = \pi$, σ_β also increases indefinitely, but has another order with respect to η. Thus, at the point $\theta = \pi$ we have

$$\sigma_\beta^* = \frac{2P\Delta_1\eta^{-3/2}}{\pi r_1}, \quad \Delta_1 = \frac{2k_2}{k_1(k_1^2+k_2^2)} - \frac{6}{k_1(k_1-k_2)^2}. \tag{18}$$

Formulas (17), (18) once more point to the strong stress concentration in the vicinity of the connector when its thickness is comparatively small.

A more detailed analysis of all the components of the stress tensor indicates that in the case $h \to 0$, σ_β increases indefinitely in every region S (including the boundaries), and σ_α increases indefinitely only at the inner points of S, and the shearing stresses strive to zero everywhere.

In conclusion, we would like to note that formulas (1), (2), (8) - (11) contain, as a particular case, the solution of the problem for a halfplane with a circular hole, when a concentrated force is applied to its rectilinear boundary above the connector. The solution of this problem is obtained by a limiting transition in the case $\alpha_2 \to 0$.

Stress concentration in a halfplane and a plane with circular holes. Two particular problems are investigated: (1) extension of a halfplane with a circular hole; (2) extension of a plane with two equal circular holes. The solution of the first problem was first obtained by G. Jeffery (Ref. 4) in trigonometric series. However, his solution contained an inaccuracy, which /274
was corrected by R. Mindlin (Ref. 5). The solution of the second problem also in the form of Fourier series (Ref. 1, 3) belongs to Ch. Ling. These solutions are not valid, if $\eta = h/r$ is small (h -- connector thickness, r -- hole radius).

In order to formulate valid solutions, the trigonometric series were transformed according to the formulas in (Ref. 2). In order to calculate the stresses, it is possible to obtain comparatively simple formulas for a sufficiently small η.

When the halfplane with a circular hole is extended, the law governing the normal stress distribution along the connector for a sufficiently small η may be determined by the following relationships

$$\sigma_\alpha/p = (\eta/2)^{1/2}\xi(1-\xi)(2+8\xi-3\xi^2) \quad (\xi = y/h);$$
$$\sigma_\beta/p = \begin{cases} (8/\eta)^{1/2}\xi & \text{for } y \neq 0; \\ (44/15)(2\eta^{1/2}) & \text{for } y = 0. \end{cases} \tag{19}$$

Here p are the tensile stresses at infinity; y is measured along the rectilinear boundary. Since these expressions represent the principal terms of the corresponding stresses, the smaller is η, the less accurate are the results derived from calculations based on formulas (19).

The following formula is obtained for the concentration coefficient:

$$k = \frac{4}{\varepsilon}\left(1 + \frac{7}{60}\varepsilon^2 + \frac{3}{1400}\varepsilon^4\right), \tag{20}$$

where $\varepsilon = \ln(1 + \eta + \sqrt{2\eta + \eta^2})$.

In calculations based on formula (20), the smaller is ε, the smaller the error will be, and if $0 < \varepsilon \leqq 1.9$, it does not exceed 2%. If $\varepsilon > 1.9 (\eta > 2.42)$, it may be assumed that the concentration coefficient practically equals three, since $k = 3.04$ in the case $\varepsilon = 1.9$.

When a plane with two equal circular holes is extended, the concentration coefficient may be determined according to the following formulas

$$\left.\begin{gathered} k_1 = A\left(\frac{\mathrm{sh}^2 \varepsilon}{\varepsilon^2} m_1 + 1\right);\ k_2 = A\left(\frac{\mathrm{sh}^2 \varepsilon}{\varepsilon^2} m^2 - 1\right);\ k_3 = A\frac{\mathrm{sh}^2 \varepsilon}{\varepsilon^2} m_3; \\ A = \frac{2\,\mathrm{ch}\,\varepsilon\,(\mathrm{ch}\,\varepsilon + 1)}{\mathrm{sh}\,2\varepsilon + 2\varepsilon}, \end{gathered}\right| \tag{21}$$

The subscript 1 corresponds to transverse extension, and the subscript 2 corresponds to longitudinal extension; the subscript 3 corresponds to unidirectional extension; $\varepsilon = \ln(1 + \eta + \sqrt{\eta + \eta^2/4})$. The constants m_s $(s = 1, 2, 3)$ may be determined from the following equation

$$\begin{gathered} m_s\varepsilon^2\left(0,716 - 0,0393\varepsilon^2 + \frac{2\,\mathrm{sh}^4 \varepsilon}{\mathrm{sh}\,2\varepsilon + 2\varepsilon}\right) = \frac{1}{2} \mp \frac{1}{2} \mp \frac{\mathrm{sh}^2 \varepsilon}{\mathrm{sh}\,2\varepsilon + 2\varepsilon} \pm \\ \pm \frac{\mathrm{sh}^2 \varepsilon}{\varepsilon^2}(0,769 - 0,179\varepsilon^2 + 0,341\varepsilon^4). \end{gathered} \tag{22}$$

In the latter formula, the superscript is chosen for the case of transverse extensions, and the subscript is chosen for the case of longitudinal extension. For unidirectional extension, the components with a double index must be omitted. /275

The error of the calculations based on formulas (21), (22), does not exceed 2% in the case $0 < \varepsilon \leqq 1$. If $\varepsilon > 1$, the Ling solution is sufficiently valid.

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NUMERICAL METHOD FOR CONFORMAL MAPPING OF SIMPLY AND MULTIPLY CONNECTED REGIONS, BASED ON TRIGONOMETRIC INTERPOLATION /276

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1. Let us investigate the problem of mapping a unit circle $|\zeta| \leq 1$ onto a simply-connected region $z = x + iy$ which is given beforehand and which is bounded by a simple closed profile. We may standardize the mapping function $z = f(\zeta)$, as is customary, by the conditions

$$z = f(\zeta)|_{\zeta=0} = 0; \quad z = f(\zeta)|_{\zeta=1} = x_0, \tag{1}$$

i.e., we require that the points $\zeta = 0$ and $\zeta = 1$ be mapped at the points $z = 0$ and $z - x_0$ (Figure 1).

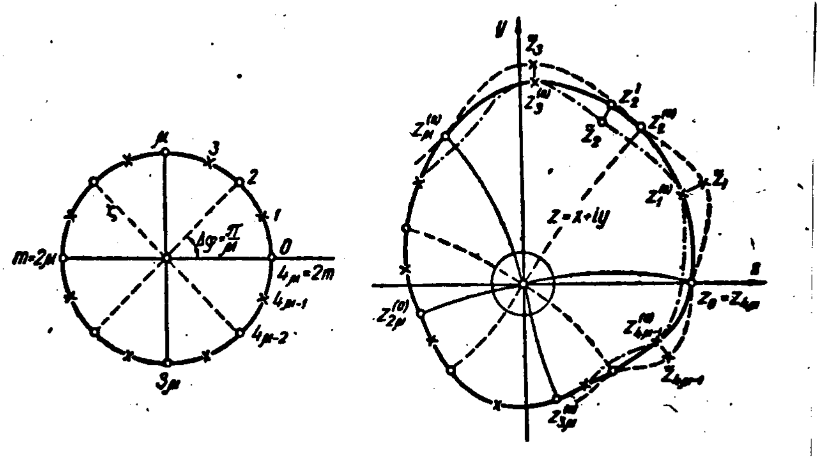

Figure 1

We shall to determine the mapping function in the form of a polynomial with complex coefficients C_n:

$$z = \sum_{n=1}^{m} C_n \zeta^n; \quad C_n = A_n + iB_n. \tag{2}$$

In the case of $m \to \infty$, the polynomial (2) is transformed into a series which will perform the precise mapping function, according to the Riemann theory.

Let us divide the unit circle $|\zeta| = 1$ into $2m = 4\mu$ equal parts, so that the polar coordinates of the division points $\zeta_n = r_n e^{i\phi_n}$ will be, respectively,

$$r_n = 1; \quad \varphi_n = \frac{n\pi}{m}; \quad n = 0, 1, 2, \ldots, 2m-1. \tag{3}$$

We shall designate the transforms of these points $z_n = x_n + iy_n$ in the region z as *nodal points* (Figure 1).

The orthoganality conditions of the trigonometric functions of the discrete argument will play a significant role below in the case of equally spaced data. An elementary proof for these data was given by A. N. Krylov /277
even in 1906 [(Ref. 2), Section 53; (Ref. 5), Section 59], namely:

$$\left.\begin{aligned}&\sum_{n=1}^{m}\sin j\varphi_n\sin\nu\varphi_n=\begin{cases}0 \text{ for } j\neq\nu;\\ m/2 \text{ for } j=\nu;\end{cases}\\&\sum_{n=1}^{m}\cos j\varphi_n\cos\nu\varphi_n=\begin{cases}0 \text{ for } j\neq\nu;\\ m/2 \text{ for } j=\nu;\end{cases}\\&\sum_{n=1}^{m}\sin j\varphi_n\cos\nu\varphi_n=0 \text{ for any } j.\end{aligned}\right\} \quad (4)$$

In the region ζ, let us now change to polar coordinates and, employing the Euler formulas for ζ^{ν},

$$\zeta^{\nu}=r^{\nu}e^{i\nu\varphi}=r^{\nu}(\cos\nu\varphi+i\sin\nu\varphi),$$

let us represent equation (2) in the following form

$$z=x+iy=\sum_{\nu=1}^{m}(A_{\nu}+iB_{\nu})\,[r^{\nu}(\cos\nu\varphi+i\sin\nu\varphi)].$$

Separating the real and imaginary parts, we obtain

$$x=\sum_{\nu=1}^{m}r^{\nu}(A_{\nu}\cos\nu\varphi-B_{\nu}\sin\nu\varphi);\quad y=\sum_{\nu=1}^{m}r^{\nu}(A_{\nu}\sin\nu\varphi+B_{\nu}\cos\nu\varphi). \quad (5)$$

In particular, for the nodal points for which $r = 1$, $\phi = \phi_n = \frac{n\pi}{m}$, we have

$$\left.\begin{aligned}x_n&=\sum_{\nu=1}^{m}A_{\nu}\cos\nu\varphi_n-B_{\nu}\sin\nu\varphi_n;\\ y_n&=\sum_{\nu=1}^{m}A_{\nu}\sin\nu\varphi_n+B_{\nu}\cos\nu\varphi_n.\end{aligned}\right\} \quad (6)$$

Equations (6) enable us to calculate the nodal points readily from the known coefficients A_{ν}, B_{ν}. However, we are also interested in the opposite problem -- determining the coefficients A_{ν}, B_{ν} from the nodal points. For this purpose, we must invert system (6), which may be regarded as a system of 2m equations with 2m unknowns A_j; B_j; $j = 1, 2, \ldots, m$. The solution of the system of linear algebraic equations for large m is a problem which is extremely complex in technical terms. However, in this case, if the angles ϕ_n are selected as equidistant, and if the orthogonality conditions (4) are employed, system (6) may be transformed quite simply, and as a result we obtain: /278

$$\left.\begin{aligned}A_j&=\frac{1}{m}\sum_{n=1}^{m}x_n\cos j\varphi_n+y_n\sin j\varphi_n;\quad \varphi_n=\frac{n\pi}{m};\\ B_j&=\frac{1}{m}\sum_{n=1}^{m}y_n\cos j\varphi_n-x_n\sin j\varphi_n;\quad j=1,2,\ldots,m.\end{aligned}\right\} \quad (7)$$

Actually, multiplying the first of the equations (6) by $\cos j\phi_n$, and the second by $\sin j\phi_n$ and summing up the results with respect to n, we obtain

$$\sum_{n=1}^{m}x_n\cos j\varphi_n+y_n\sin j\varphi_n=\sum_{n=1}^{m}\left\{\sum_{\nu=1}^{m}(A_{\nu}\cos\nu\varphi_n-B_{\nu}\sin\nu\varphi_n)\cos j\varphi_n+\right.$$

$$+\sum_{\nu=1}^{m}(A_\nu \sin\nu\varphi_n + B_\nu \cos\nu\varphi_n)\sin j\varphi_n\} = \sum_{\nu=1}^{m} A_\nu \{\sum_{n=1}^{m}\cos j\varphi_n \cos\nu\varphi_n +$$

$$+\sum_{n=1}^{m}\sin j\varphi_n \sin\nu\varphi_n\} + \sum_{\nu=1}^{m} B_\nu \{\sum_{n=1}^{m}\sin j\varphi_n\cos\nu\varphi_n - \sum_{n=1}^{m}\cos j\varphi_n \sin\nu\varphi_n\} =$$

$$= A_j\left(\frac{m}{2}+\frac{m}{2}\right) = mA_j,$$

since all the inner sums, with the exception of only the two sums with respect to $A_\nu = A_j$, vanish in view of the orthogonality conditions (4). We may obtain the formula for B_j in a similar manner.

2. Formulas (7) reduce the problem of calculating the coefficients of the mapping function (2) to determining the nodal points which are not known at the beginning. Let us now formulate the iteration process for determining the nodal points within an accuracy which is determined beforehand. This represents the basic difficulty of this problem.

For this purpose, let us investigate two systems of m nodal points: even $n = 2\nu$ and odd $n = k = 2\nu - 1$, $\nu = 1, 2, \ldots, m$.

The coefficients A_j, B_j, formulated according to a system of m even nodal points, may be designated by $A_j^{(+m)}$, $B_j^{(+m)}$, and by $A_j^{(-m)}$, $B_j^{(-m)}$ according to odd nodal points, so that we shall have the following, according to (7) /279

$$A_j^{(+m)} = \frac{1}{m}\sum_{\nu=1}^{m} x_{2\nu}\cos j\varphi_{2\nu} + y_{2\nu}\sin j\varphi_{2\nu}; \quad j = 1, 2, \ldots, m; \tag{8}$$

$$B_j^{(+m)} = \frac{1}{m}\sum_{\nu=1}^{m} y_{2\nu}\cos j\varphi_{2\nu} - x_{2\nu}\sin j\varphi_{2\nu}; \quad z_{2m} = z_0$$

and correspondingly

$$A_j^{(-m)} = \frac{1}{m}\sum_{k=1}^{2m-1} x_k\cos j\varphi_k + y_k \sin j\varphi_k; \quad j = 1, 2, \ldots, m; \tag{9}$$

$$B_j^{(-m)} = \frac{1}{m}\sum_{k=1}^{2m-1} y_k\cos j\varphi_k - x_k\sin j\varphi_k; \ k = 1, 3, 5, \ldots, 2m-1.$$

The following formulas hold for the quantities introduced (Ref. 4):

$$A_j^{(\pm m)} = \sum_{\nu=0}^{\infty}(\pm 1)^\nu a_{j+\nu m} = a_j \pm a_{j+m} + a_{j+2m} \pm \ldots; \tag{10}$$

$$B_j^{(\pm m)} = \sum_{\nu=0}^{\infty}(\pm 1)^\nu b_{j+\nu m} = b_j \pm b_{j+m} + b_{j+2m} \pm \ldots,$$

where a_n, b_n are the coefficients of the series representing the precise mapping function, i.e., the series into which the polynomial (2) changes in the case $m \to \infty$.

Let us assume that for given $m = 2\mu$ the values of the even nodal points are known in the zero approximation

$$z_{2\nu}^{(0)} = x_{2\nu}^{(0)} + iy_{2\nu}^{(0)}; \quad \nu = 1,\ 2,\ \ldots,\ m; \quad z_{2m} = z_0. \tag{11}$$

This enables us to employ formulas (8) to compute all $A_j^{(+m)}$, $B_j^{(+m)}$; $j = 1, 2, \ldots, m$, and the corresponding polynomial of the $m\underline{th}$ power (2)

$$\tilde{z}^{(0)} = P_m(\zeta) = C_1^{(m)}\zeta + C_2^{(m)}\zeta^2 + \ldots + C_m^{(m)}\zeta^m;$$
$$C_j^{(m)} = A_j^{(+m)} + iB_j^{(+m)} \tag{12}$$

maps the unit circle $|\zeta| \leq 1$ onto the region $\tilde{z}^0$, so that its boundary will precisely pass through the given nodal points (11). This statement follows from the derivation itself of formulas (8), since the quantities $A_j^{(+m)}$, $B_j^{(+m)}$ were determined from a system of linear algebraic equations, whose right hand sides contain the coordinates of the given nodal points $x_{2\nu}^{(0)}$, $y_{2\nu}^{(0)}$.

If we now assume approximately the following /280

$$A_j^{(+m)} \approx a_j^{(0)}; \quad B_j^{(+m)} \approx b_j^{(0)}, \tag{13}$$

we may calculate the odd points $\tilde{z}_k = \tilde{x}_k + iy_k$ from these coefficients according to formulas (6). These points, generally speaking, will not lie on the boundary of the given region z (see Figure 1).

Let us plot them on the normal to the profile or by any other method (Ref. 5). As a result, we obtain a system of m odd nodal points

$$z_k^{(0)} = x_k^{(0)} + iy_k^{(0)}; \quad k = 1,\ 3,\ 5,\ \ldots,\ 2m-1, \tag{14}$$

according to which, according to formula (9), we may calculate all $C_j^{(-m)} = A_j^{(-m)} + iB_j^{(-m)}$.

The new polynomial

$$\tilde{z}^{(1)} = P_{-m}(\zeta) = C_1^{(-m)}\zeta + C_2^{(-m)}\zeta^2 + \cdots + C_m^{(-m)}\zeta^m \tag{15}$$

maps the unit circle $|\zeta| \leq 1$ onto the region $\tilde{z}^{(1)}$, so that the boundary of the region $\tilde{z}^{(1)}$ will now precisely pass through the odd points (14) lying on the given profile. Therefore, setting

$$A_j^{(-m)} \approx a_j^{(1)}; \quad B_j^{(-m)} \approx b_j^{(1)}, \tag{16}$$

by employing the same formulas (6) we may calculate the approximate points $\tilde{z}_{2\nu} = \tilde{x}_{2\nu} + iy_{2\nu}$; $\nu = 1, 2, \ldots, m$, and, plotting them on the given profile, we obtain a more accurate value of the desired even nodal points (see Figure 1).

We may repeat the iteration process until the subsequent and preceding approximations coincide within the given accuracy. Thus, employing the coefficients (13) or (16), we may calculate an arbitrary number of points, in particular 2m, 4m or $2^\nu m$, which enables us to double, quadruple, or increase by a factor of 2^ν the magnitude of the iteration cycle m. For sufficiently large $m = 2^\nu$, the quantities $A_j^{(+m)}$, $A_j^{(-m)}$ and $B_j^{(+m)}$, $B_j^{(-m)}$ coincide within any degree of accuracy which is specified beforhand, which is a signal that the process has ended.

Actually, in view of the Riemann theorem, the series obtained from (2) in the case $m \to \infty$ is a converging series. Therefore, for any ε which is specified beforehand, we may always determine the number m such that the following conditions are satisfied

$$|\pm a_{j+m}+a_{j+2m}\pm\ldots|<\frac{\varepsilon}{2};\quad |\pm b_{j+m}+b_{j+2m}\pm\ldots|<\frac{\varepsilon}{2}, \tag{17}$$

Also, according to equations (10), within an accuracy of ξ we shall have

$$A_j^{(+m)}\doteq A_j^{(-m)}=a_j;\quad B_j^{(+m)}=B_j^{(-m)}=b_j;\ j=1,\ 2,\ \ldots,\ m. \tag{18}$$

We may determine the initial values of $z_{2\nu}^{(0)}$ graphically [(Ref. 5, section 631)] or by means of electromodeling (Ref. 3).

3. The iteration process described in section 2 may be significantly simplified if we obtain formulas which directly relate the even and odd nodal points. The necessity is thus eliminated of calculating at each step the intermediate approximate coefficients $A_j^{(\pm m)}$; $B_j^{(\pm m)}$, since the entire iteration process will be immediately satisfied by the nodal points. Determining the nodal points within the requisite degree of accuracy, we may calculate only once the desired final coefficients $A_j^{(+m)} = A_j^{(-m)} = a_j$; $B_j^{(+m)} = B_j^{(-m)} = b_j$ according to formulas (8) or (9). /281

Let us now derive the requisite formulas, and let us express the k-points — i.e., the odd nodal points -- by means of the even 2ν-points.

Substituting the coefficients $A_j^{(+m)}$ and $B_j^{(+m)}$ in formulas (6) for the odd $n = k = 2\nu - 1$, we have

$$x_k=\sum_{j=1}^{m}A_j^{(+m)}\cos j\varphi_k-\underset{(+m)}{B_j^{(+m)}}\sin \underset{(+m)}{j\varphi_k};$$
$$y_k=\sum_{j=1}^{m}A_j^{(+m)}\sin j\varphi_k+B_j^{(+m)}\cos j\varphi_k.$$

Thus, omitting A_j, B_j by means of formulas (8), we obtain

$$x_k=\frac{1}{m}\sum_{j=1}^{m}\sum_{\nu=1}^{m}(x_{2\nu}\cos j\varphi_{2\nu}+y_{2\nu}\sin j\varphi_{2\nu})\cos j\varphi_k+$$
$$+(x_{2\nu}\sin\varphi_{2\nu}-y_{2\nu}\cos j\varphi_{2\nu})\sin j\varphi_k=\frac{1}{m}\sum_{j=1}^{m}\sum_{\nu=1}^{m}x_{2\nu}\cos j(\varphi_{2\nu}-\varphi_k)+$$
$$+y_{2\nu}\sin j(\varphi_{2\nu}-\varphi_k),$$

or, taking (3) into account, we obtain

$$x_k=\frac{1}{m}\sum_{j=1}^{m}\sum_{\nu=1}^{m}x_{2\nu}\cos j\varphi_{2\nu-k}+y_{2\nu}\sin j\varphi_{2\nu-k};$$
$$y_k=\frac{1}{m}\sum_{j=1}^{m}\sum_{\nu=1}^{m}y_{2\nu}\cos j\varphi_{2\nu-k}-x_{2\nu}\sin j\varphi_{2\nu-k}, \tag{19}$$

where

$$\varphi_{2\nu-k} = \varphi_{2\nu} - \varphi_k = \frac{(2\nu-k)\pi}{m}; \quad \nu = 1,\ 2,\ \ldots,\ m;$$
$$k = 1,\ 3,\ 5,\ \ldots,\ 2m-1.$$

In addition, since

$$(m-j)\,\varphi_{2\nu-k} = \frac{(m-j)(2\nu-k)\pi}{m} = (2\nu-k)\pi - j\varphi_{2\nu-k},$$ /282

TABLE 1

$2\nu-k$	$m=4$	$m=8$	$m=16$	$m=32$	$2\nu-k$
	$\gamma_{2\nu-k}$	$\gamma_{2\nu-k}$	$\gamma_{2\nu-k}$	$\gamma_{2\nu-k}$	
1	0,603 553 39	0,628 417 44	0,634 573 15	0,636 108 36	1
3	0,103 553 39	0,187 075 72	0,206 034 89	0,210 670 39	3
5		0,083 522 33	0,116 929 28	0,124 756 99	5
7		0,024 864 05	0,076 156 47	0,087 337 90	7
9			0,051 292 42	0,066 072 57	9
11			0,033 406 95	0,052 137 48	11
13			0,018 959 17	0,042 135 75	13
15			0,006 155 71	0,034 479 06	15
17				0,028 323 35	17
19				0,023 176 58	19
21				0,018 730 53	21
23				0,014 780 15	23
25				0,011 181 43	25
27				0,007 827 72	27
29				0,004 635 50	29
31				0,001 535 21	31

we have the following, taking the fact into account that the number $2\nu - k$ is always odd

$$\cos j\varphi_{2\nu-k} + \cos(m-j)\,\varphi_{2\nu-k} = 0; \quad \cos\frac{m}{2}\varphi_{2\nu-k} = 0;$$
$$\cos m\varphi_{2\nu-k} = \cos(2\nu-k)\pi = \cos\pi = -1.$$

In this case, for any m; $2\nu = 2, 4 \ldots, 2m$; $k = 1, 3 \ldots, 2m - 1$, we have

$$\sum_{j=1}^{m} \cos j\varphi_{2\nu-k} = \cos\frac{m}{2}\varphi_{2\nu-k} + \cos m\varphi_{2\nu-k} + $$
$$+ \sum_{j=1}^{m/2-1} [\cos j\varphi_{2\nu-k} + \cos(m-j)\,\varphi_{2\nu-k}] = -1.$$

Substituting these values in (19) and changing the order of summation, we obtain

$$\left.\begin{aligned} x_k &= \frac{-1}{m}\sum_{\nu=1}^{m} x_{2\nu} + \frac{1}{m}\sum_{\nu=1}^{m}\sum_{j=1}^{m} y_{2\nu}\sin j\varphi_{2\nu-k}; \\ y_k &= \frac{-1}{m}\sum_{\nu=1}^{m} y_{2\nu} + \frac{1}{m}\sum_{\nu=1}^{m}\sum_{j=1}^{m} x_{2\nu}\sin j\varphi_{2\nu-k} \end{aligned}\right\} \quad (20)$$

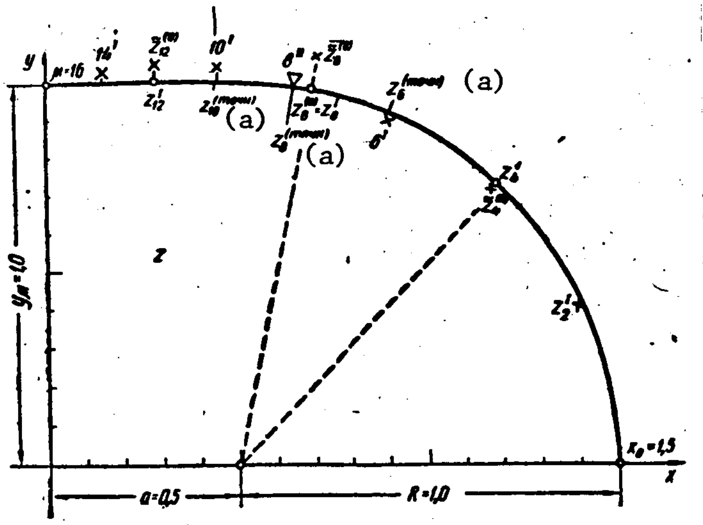

Figure 2
(a) - Point.

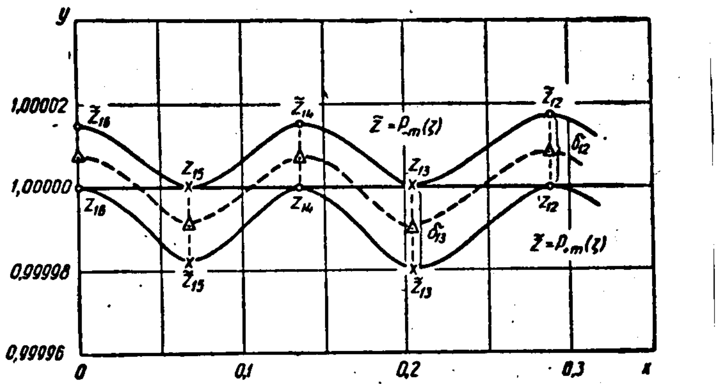

Figure 3

or finally

$$x_k = \sigma_x + \sum_{\nu=1}^{m} y_{2\nu}\gamma_{2\nu-k}; \quad y_k = \sigma_y - \sum_{\nu=1}^{m} x_{2\nu}\gamma_{2\nu-k}, \tag{21}$$

where

$$\gamma_{2\nu-k} = \frac{1}{m}\sum_{j=1}^{m} \sin j\varphi_{2\nu-k} = \frac{1}{m}\operatorname{ctg}\frac{1}{2}\varphi_{2\nu-k}; \quad \varphi_{2\nu-k} = \frac{(2\nu-k)\pi}{m}; \tag{22}$$

$$\sigma_x = \frac{-1}{m}\sum_{\nu=1}^{m} x_{2\nu}; \quad \sigma_y = \frac{-1}{m}\sum_{\nu=1}^{m} y_{2\nu}. \tag{23}$$

Exchanging the roles of the numbers 2ν and k in (19) and taking the fact into account $\sin\phi_{2\nu-k} = -\sin\phi_{k-2\nu}$, and $\cos\phi_{2\nu-k} = +\cos\phi_{k-2\nu}$, we obtain the formulas for directly computing the even nodal points from the odd nodal

TABLE 2

k	x_k^V	y_k^V	2ν	$\tilde{x}_{2\nu}^{VI}$
1	1,476 618	0,214 983	0	1,500 052
3	1,307 548	0,589 802	2	1,409 688
5	1,043 827	0,839 197	4	1,181 805
7	0,768 660	0,963 235	6	0,903 756
9	0,531 995	0,999 488	8	0,643 750
11	0,353 268	1,000 000	10	0,436 547
13	0,203 945	1,000 000	12	0,276 286
15	0,066 794	1,000 000	14	0,134 426
			16	0

$\tilde{y}_{2\nu}^{VI}$	$\delta_{2\nu}$	$x_{2\nu}^{VI}$	$y_{2\nu}^{VI}$
0	0,000 052	1,500 000	0
0,415 409	0,000 048	1,409 644	0,415 389
0,731 595	0,000 044	1,181 775	0,731 562
0,914 909	0,000 038	0,903 740	0,914 874
0,989 642	0,000 027	0,643 746	0,989 615
1,000 022	0,000 022	0,436 547	1,000 000
1,000 017	0,000 017	0,276 286	1,000 000
1,000 015	0,000 015	0,134 426	1,000 000
1,000 015	0,000 015	0	1,000 000

points in absolutely the same manner:

$$x_{2\nu} = \sigma'_x - \sum_{k=1}^{2m-1} y_k \gamma_{2\nu-k}; \quad y_{2\nu} = \sigma'_y - \sum_{k=1}^{2m-1} x_k \gamma_{2\nu-k}, \tag{24}$$

where

$$\sigma'_x = \frac{-1}{m} \sum_{k=1}^{2m-1} x_k; \quad \sigma'_y = \frac{-1}{m} \sum_{k=1}^{2m-1} y_k; \quad k = 1, 3, \ldots, 2m-1, \tag{25}$$

and the coefficients $\gamma_{2\nu-k}$ retain their previous value (22).

We should stress that all $\gamma_{2\nu-k}$ are constant quantities and do not depend /283 on the form of the mapping region, so that they may be calculated once and for all. Thus, for given m different basis values of $\gamma_{2\nu-k}$, $\mu = \frac{m}{2}$ only will hold, since according to (22) we have

$$\gamma_{-n} = -\gamma_n; \quad \gamma_{2m\pm n} = \pm \gamma_n. \tag{26}$$

Table 1 presents the basis values of the coefficients $\gamma_{2\nu-k}$ for m = 4, 8, 16, 32 computed to eight decimal places. All of the computed formulas assume a simpler form for the symmetrical regions.

Example. Within an accuracy of $|\delta| \leq 0.00005$, let us map a unit circle $|\zeta| \leq 1$ onto the region z bounded by a box-like curve, i.e., onto a rounded rectangle, whose dimensions are shown in Figure 2.

TABLE 3 /284

j	$A_j^{(+32)}$	$A_j^{(-32)}$	j	$A_j^{(+32)}$	$A_j^{(-32)}$
1	+1,201 333	+1,201 361	17	+0,000 253	+0,000 254
3	0,246 170	0,246 188	19	0,000 016	0,000 016
5	0,047 952	0,047 957	21	—0,000 142	—0,000 141
7	0,002 871	0,002 872	23	0,000 059	0,000 060
9	0,000 840	0,000 841	25	0,000 054	0,000 053
11	0,001 164	0,001 165	27	—0,000 068	—0,000 069
13	—0,000 298	—0,000 299	29	0,000 008	0,000 009
15	—0,000 254	—0,000 254	31	0,000 040	0,000 040

Starting with the given $x_0 = 1.5$ and $y_\mu = 1.0$, determining the even nodal points successively in the case m = 4, 8, 16 in the zero approximation, as is shown by the crosses in Figure 2, and then performing three steps on the level m = 16 and two steps on the level m = 32, we may solve the formulated problem. Table 2 presents all the computations for the second half of the fifth step (n = V), the deviation from the profile $\delta_{2\nu}$, of the approximate nodal points $\tilde{z}_{2\nu}$. Table 3 presents all the coefficients $A_j^{(\pm m)}$, computed on the basis of the even and odd nodal points which are found, in the case m = 32, and $A_2 = A_4 = \ldots$ $\ldots = A_m = 0$.

In the case of symmetrical regions, all the coefficients $B_j^{(\pm m)} = 0$. Figure 3 presents the approximating polynomials $\tilde{z} = P \pm m\ (\zeta)$ found for m = 32 in the section $0 \leq x \leq 0.30$. For greater clarity, the scale on the y-axis is increased by a factor of 2500.

A more detailed treatment of the computational method and further examples are presented in the studies (Ref. 4, II, and 5 chapter III, § 59-63). /285

4. The method presented above may be readily generalized to the case of external and doubly-connected regions. Thus, for external regions the given accuracy is achieved, as a rule, for considerably smaller m, so that the amount of calculations is much less than for mapping of the same inner regions. In the case of doubly-connected regions, we may formulate the iteration process similarly to the manner employed in the articles by B. F. Shilov (Ref. 6) and Yu. V. Blagoveshchenskiy (Ref. 1). Employing the results given in section 3, we may establish the direct relationship between the even and odd nodal points.

We may formulate another method, which may be applied to the mapping of regions having any connectivity, by means of the method of successive conformal mapping. For this purpose, we may first reduce the n-connected region -- drawing the n - 1 branch cuts in it -- to a simply-connected region. By means of the method of successive conformal mappings (or combining it with the method /286 of trigonometrec interpolation), we may map this region onto a halfplane with obligatory specification of the transforms of all the branch cuts. All the transforms of these branch cuts will be located on the real axis. If we then map the halfplane with a n - 1 branch cut on the real axis onto the corresponding canonical region, so that the branch cut transforms change into the corresponding branch cuts reducing the canonical region to a simply-connected region, we may formulate the requisite mapping. Figure 4 schematically shows /287 the mapping of a simply-connected region onto a halfplane with a horizontal branch cut, which may be most conveniently used as the canonical region when solving the problems of hydroaeromechanics. Thus, in the first step (without drawing the branch cut) we may map the deformed halfplane z onto the halfplane z_1, and may simultaneously calculate the series of the transforms of points on the contour Γ_2, according to which we may formulate this profile in the region z_1. After the line Γ_1 is mapped onto the real axis x_1, we may draw the branch cut ABCDE and may employ the function E_s to map the simply-connected region z_1 which is obtained onto the region z_2, and then onto the halfplane ζ,

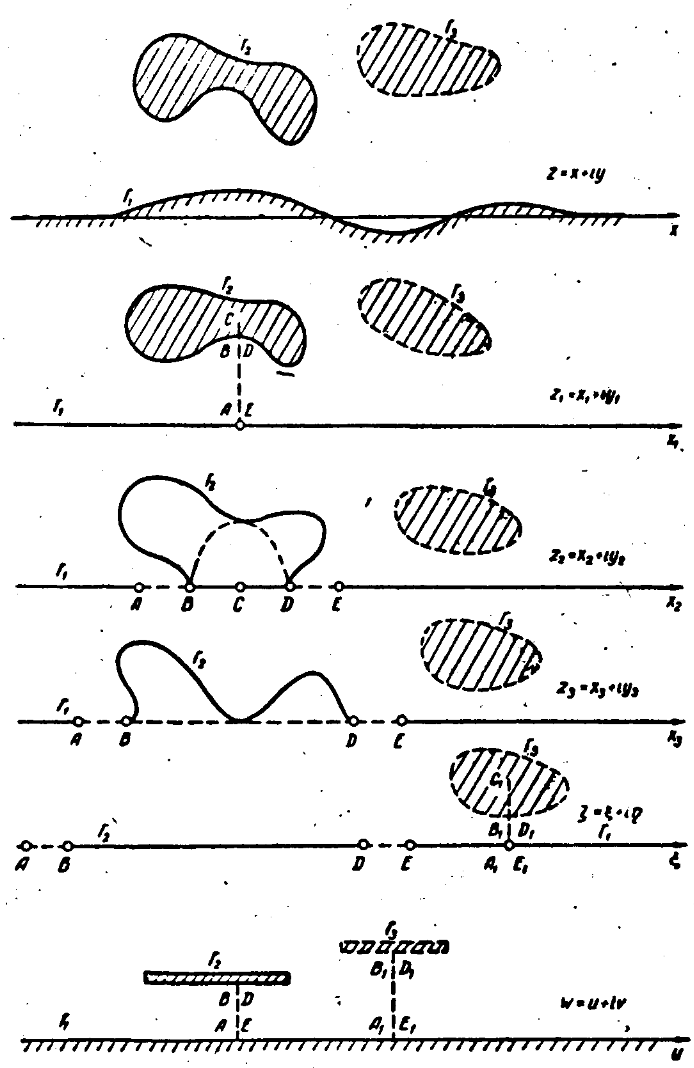

Figure 4

as was discussed in detail in [Ref. 5, chapter III, sections 51-55]. When the canonical region w is mapped onto the halfplane ζ, it is advantageous to employ the Christoffel-Schwarz integral. The method for determining the constants of this integral was also discussed in [Ref. 5, chapter III, § 51-55]. The dashed lines in Figure 4 show the manner in which the results are generalized to the case of triply-connected regions. In particular, this method yields good results when solving the problem of an underwater wing in a shoal, when the bottom has an arbitrary form. When solving the problems of elasticity theory, it is advantageous to select a concentric ring with concentric branch cuts (in the case $n \geq 3$) as the canonical region w.

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COMPLETE ELIMINATION OF STRESS CONCENTRATIONS AROUND HOLES IN PLATES /288

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N67-24538

The problem of the bending of thin plates with holes has been investigated in studies by several authors. Only two conditions are satisfied out of the three natural boundary conditions on the hole profiles in these studies, as well as in the elementary theory of plate bending in general. Due to this fact, the problem regularly arises of the reliability of the stress concentration coefficient thus determined around holes. Actually, the transverse shearing stresses, which should equal zero on the free edge of the hole, do not equal zero. Their influence on the plate deformation thus increases with a decrease in the hole dimensions, and we cannot overlook them, as compared with the influence of bending moments (Ref. 6).

We can raise a similar question with respect to the problems of total elimination of stress concentration by equivalent reinforcement of the holes when thin plates are bent (Ref. 7, 5). This is due to the fact that only two boundary conditions are satisfied here on the profile of the plate joint with an elastic rib. It is therefore natural to make an attempt to clarify these problems, based on special plate theories which enable us to satisfy all three boundary conditions on the hole profiles.

The problem of the accuracy of the stress concentration coefficient obtained by the classical theory of thin plate bending has already been studied by several authors (Ref. 6, et al.).

However, we should point out that the main problem in designing laminated objects with holes is not, as is known, determining the stress concentration, but the elimination of this concentration. This article is devoted to this /289
problem.

Let us investigate a finite elastic plate with curvilinear holes which are bounded by the profiles L'_k $(k = 1, 2, ..., n)$. The hole edges are reinforced by elastic curvilinear rods having variable rigidity, which are joined to the plate. It is assumed that the reinforcing elements are thin, and therefore the conditions under which they are joined to the plate are not investigated at the profiles L'_k, but on their axial lines L_k. The rod axes and one of the main central axes of inertia of their transverse cross-sections lie in the middle plane of the plate.

We shall call such a rod-like system (ring) an equivalent reinforcing system of the kth hole of the plate. For a given loading, this system completely replaces the influence of the absent portion of the plate within L_k, i.e., it specifies that the stress-deformed states of a solid plate without a hole and a plate with a hole, reinforced by an equivalent ring (the condition of equivalence), are identical.

By definition, when solving the problem of the equivalent reinforcement of a plate with n holes, without restricting the generality we may confine ourselves to investigating the case n = 1.

In different variations of special theories [see (Ref. 1, 2, 9) et al.], the plate bending is described by a system of differential equations of the sixth order with respect to three independent quantities, namely the normal bending of the middle plane w(x, y) and two functions which are introduced in various ways by different authors.

In order to solve the boundary value problems, we believe that a system of differential equations of plate bending is expedient, in which the normal bending w and the elastic rotation angles of the normal element γ_x and γ_y, pertaining to the middle plane, are included as the desired functions. This system was obtained by B. F. Vlasov (Ref. 2). It was also obtained as a particular case, out of the more general equations encompassing different variations specifying the conditions on the boundary equivalent planes, by M. P. Sheremet'yev and B. L. Pelekh (Ref. 8). If the external stresses on the bases of the plate equal zero, then these equations have the following form (Ref. 8)

$$\left.\begin{aligned} &\Delta w+\frac{\partial\gamma_x}{\partial x}+\frac{\partial\gamma_y}{\partial y}=0;\\ &\Delta\gamma_x-\frac{5}{2}h^{-2}\gamma_x=\frac{5}{2}h^{-2}\frac{\partial w}{\partial x}+\frac{3+2\nu}{2(1-\nu)}\cdot\frac{\partial}{\partial x}(\Delta w);\\ &\Delta\gamma_y-\frac{5}{2}h^{-2}\gamma_y=\frac{5}{2}h^{-2}\frac{\partial w}{\partial y}+\frac{3+2\nu}{2(1-\nu)}\cdot\frac{\partial}{\partial y}(\Delta w), \end{aligned}\right\} \tag{1}$$

where Δ is the Laplace operator; 2h -- plate thickness; ν -- Poisson coefficient.

By means of the functions w, γ_x, γ_y, we may write the conditions of equivalence in the following form /290

$$\left.\begin{aligned} &w(x, y)=w^{0}(x, y);\\ &\gamma_x(x, y)=\gamma_x^{0}(x, y);\\ &\gamma_y(x, y)=\gamma_y^{0}(x, y), \end{aligned}\right\} \tag{2}$$

where the quantities on the right are the main bendings and rotation angles for the solid plate, and the quantities on the left are the bendings and rotation angles for a plate with a reinforced hole under the same loading.

The junction conditions at L_1 between the plate and the equivalent ring have the following form

$$p(s)=N_n;\quad h(s)=H_{n\tau};\qquad m(s)=M_n; \tag{3}$$

$$w_p=w;\quad \gamma_1(s)=-\gamma_n(s);\ \gamma_2(s)=\gamma_s(s), \tag{4}$$

where p(s), h(s), m(s) are the transverse forces, the bending moments, and the twisting moments influencing the ring from the plate; w_p, $\gamma_1(s)$, $\gamma_2(s)$ -- bending, twisting angle, and bending angle of the ring; N_n, $H_{n\tau}$, M_n -- intersection force, torque, and bending moment in the plate at L_1; γ_s, γ_n -- pertain to the middle plane of the rotation angle for the normal element around the main normal $\vec{n}$ and the tangent $\vec{\tau}$ to L_1.

The right hand sides of conditions (3) and (4) are known, since only the corresponding problem for the solid plate will be solved. In the general case, the quantities γ_n and γ_s may be expressed by the following formulas (Ref. 8):

$$\gamma_n = -\frac{\partial w}{\partial n} + e_{nz};\ \gamma_s = -\frac{\partial w}{\partial s} + e_{sz}, \tag{5}$$

in which e_{nz} and e_{sz} are displacements pertaining to the middle plane of the plate.

The problem of selecting the equivalent ring consists of determining its rigidity according to the given loading (3) and deformation (4). We have the following from the Clebsch relationships (Ref. 4) for a thin plane ring

$$\delta p = \frac{d\gamma_2}{ds} + q\gamma_1;\quad \delta r = \frac{d\gamma_1}{ds} - q\gamma_2, \tag{6}$$

where δ_p and δ_r are the increases in curvature and twisting of the ring; q -- variable curvature of the ring axis; s -- arc along L_1.

Comparing formulas (5) and (4), we find that, if the displacements differ from zero in a solid plate, we may obtain the solution for the problem only by utilizing the deformation theory of reinforced rods which takes the fact into account that their transverse cross-sections do not remain normal to the curved axis, in general.

Adopting this hypothesis, we may add one relationship to (6) for the /291
displacement e_p of a reinforcing rib.

$$e_p = \frac{dw_p}{ds} + \gamma_2. \tag{7}$$

We may write the dependences between the internal stresses in the ring and the deformation parameters in the following form

$$L_n = A\delta p;\ L_\tau = C\delta r;\ V_B = Be_p, \tag{8}$$

where L_n is the bending moment, L_τ -- torque, V_B -- intersection force; A, C -- the desired ring rigidities in bending and in twisting; B -- ring rigidity for displacement.

The quantities L_n, L_τ and V_B may be determined by integrating the equations of statics for the portion of the ring subjected to loading (Ref. 3).

Taking into account the junction conditions at L_1 (4) and formulas (5) and (7), we may express the desired ring rigidities from (8) in the following form

$$A(s) = \frac{L_n}{\frac{d\gamma_s}{ds} - q\gamma_n};\quad C(s) = \frac{L_\tau}{-\frac{d\gamma_n}{ds} - q\gamma_s};\quad B(s) = \frac{V_B}{\gamma_s + \frac{dw}{ds}}. \tag{9}$$

The existence of solution (9) must be studied in each particular case.

When there are displacements in the plate, the rigidities of the equivalent ring in bending and in twisting differ, generally speaking, from the corresponding rigidities obtained within the framework of the classical bending theory (Ref. 7). In addition, this approach enables us to find one third of the rigidity for displacements, and to determine more accurately the parameters of the reinforcing element.

Let us investigate a rectangular plate having finite dimensions, loaded at the edges by bending moments M_x, M_y which are uniformly distributed and by torques H_{xy} (the x, y-axes are directed along the plate axes of symmetry). The solution of equation (1) for this case has the following form

$$\left.\begin{aligned} w &= \frac{1}{2D(1-\nu^2)}[(\nu M_y - M_x)x^2 + (\nu M_x - M_y)y^2 - \\ &\quad - 2(1+\nu)H_{xy}xy]; \\ \gamma_x &= \frac{1}{D(1-\nu^2)}[(M_x - \nu M_y)x + (1+\nu)H_{xy}y]; \\ \gamma_y &= \frac{1}{D(1-\nu^2)}[(M_y - \nu M_x)y + (1+\nu)H_{xy}x]. \end{aligned}\right\} \qquad (10)$$

We may employ formulas (5) to establish the fact that in this case the displacements equal zero, and the same internal stresses and moments in the plate are obtained as in the elementary theory of plate bending. In particular, $V_B = 0$.

We find from (9) that displacement rigidity B is arbitrary, and we obtain the following for the rigidities A and C in the case $e_{nz} = e_{sz} = 0$: /292

$$A(s) = \frac{L_n}{q\frac{\partial w}{\partial n} - \frac{\partial^2 w}{\partial s^2}}; \quad C(s) = \frac{L_\tau}{q\frac{\partial w}{\partial s} + \frac{\partial^2 w}{\partial n \partial s}}. \qquad (11)$$

Formulas (11) coincide in this case with similar formulas obtained in (Ref. 7). It thus follows that the rigidities A and C, found in (Ref. 7) on the basis of the elementary theory for several cases of equivalent reinforcement of holes having a different form in an isotropic rectangular plate, are correct from the aspect of a more correct theory of plate bending (Ref. 2, 8).

As we have already stated, the amount and type of ring rigidities to be determined depend on the manner in which the ring deformations are described by theory.

Let us now assume that a reinforcing ring is a thin-walled, plane curvilinear rod with a small initial curvature, whose largest transverse cross-section dimension is small as compared with the radius of curvature of the rod axis (Ref. 3).

Let the hole in the plate be circular, having the radius R. In order to determine the rigidity in bending A = EI, the rigidity in twisting $C = GI_\alpha$, and the sectorial rigidity of warping $B_\omega = EI_\omega$, just as previously, we obtain the following for the problem (10):

$$A(s) = \frac{L_n}{\frac{1}{R}\cdot\frac{\partial w}{\partial n} - \frac{\partial^2 w}{\partial s^2}}; \quad B_\omega(s) = \frac{B}{\frac{\partial^3 w}{\partial n \partial s^2} + \frac{1}{R}\cdot\frac{\partial^2 w}{\partial s^2}};$$

$$C(s) = \frac{L_\tau\left(\frac{\partial^3 w}{\partial n \partial s^2} + \frac{1}{R}\cdot\frac{\partial^2 w}{\partial s^2}\right) + B\left(\frac{\partial^4 w}{\partial n \partial s^3} + \frac{1}{R}\cdot\frac{\partial^3 w}{\partial s^3}\right)}{\left(\frac{\partial^3 w}{\partial n \partial s^2} + \frac{1}{R}\cdot\frac{\partial^2 w}{\partial s^2}\right)\left(\frac{\partial^2 w}{\partial n \partial s} + \frac{1}{R}\cdot\frac{\partial w}{\partial s}\right)}. \tag{12}$$

Here B is the bimoment which is in operation in the ring transverse cross-section and which may be determined by integrating the equations of bimoments (Ref. 3), which may be reduced to the following form

$$B' - \frac{\frac{\partial^4 w}{\partial n \partial s^3} + \frac{1}{R}\cdot\frac{\partial^3 w}{\partial s^3}}{\frac{\partial^2 w}{\partial n \partial s} + \frac{1}{R}\cdot\frac{\partial w}{\partial s}} B = H' - \frac{\frac{\partial^3 w}{\partial n \partial s^2} + \frac{1}{R}\cdot\frac{\partial^2 w}{\partial s^2}}{\frac{\partial^2 w}{\partial n \partial s} + \frac{1}{R}\cdot\frac{\partial w}{\partial s}} H.$$

For the particular cases under consideration of a uniform stress state (bending and twisting), we obtain

$$B'' + \frac{4}{R^2}B = 0.$$

We thus have /293

$$B = C_1\cos 2\theta + C_2\sin 2\theta,$$

where C_1 and C_2 are arbitrary integration constants.

Assuming that $C_1 = C_2 = 0$, we have $B = 0$. In this case, the expressions for rigidity in bending and twisting of a reinforced circular rod coincide with formulas (11), and the sectorial rigidity of warping equals zero. In the axi-symmetric case, the quantities C and B_ω are arbitrary, and $A = (1 + \nu)$ RD.

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EFFECT OF A STEADY THERMAL FIELD ON THE STRESS CONCENTRATION IN AN INFINITE ELASTIC PLANE WITH CIRCULAR HOLE

/294

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Let us investigate the problem of the stress state of an infinite elastic plane with a circular hole having the radius R in the presence of a steady thermal field T(r, θ) which satisfies the condition of boundedness at infinity and the boundary conditions on the hole profile T = f(θ). We shall thus assume that there are no external stresses on the profile, and the stresses at infinity equal zero. We may solve this problem by employing the well known analogy of N. I. Muskhelishvili between temperature stresses and stresses arising due to dislocations[1] (Ref. 1). It has the following form in complex potentials of Kolosov-Muskhelishvili:

$$\Phi(z) = \frac{\gamma}{z}, \quad \Psi(z) = \frac{\bar{\gamma}}{z} + 2R^2\frac{\gamma}{z^3}, \tag{1}$$

where $z = x + iy = re^{i\theta}$;

$$\gamma = -\frac{1}{\varkappa+1}\cdot\frac{\nu\mu}{\lambda+\mu}\frac{R}{\pi}\int_0^{2\pi} f(\theta)\, e^{i\theta} d\theta \tag{2}$$

in the problem of plane deformation. For the case of the plane stress state, i.e., for a plate, the Lamé coefficients λ,μ -- $\varkappa = 3 - 4\sigma$, $\nu = \frac{\alpha E}{1 - 2\sigma}$ (ν -- constant given by the Duhamel-Neumann law, α -- coefficient of linear expansion) must be replaced by the corresponding quantities with asterisks -- i.e., by mechanical and thermal constants in the case of the plane stress state. If we change to the Young's modulus E and the Poisson coefficient σ, the expression for γ assumes the following form in the case of plane deformation /295

$$\gamma = -\frac{\alpha E}{4\pi(1-\sigma)} R\int_0^{2\pi} f(\theta)\, e^{i\theta} d\theta, \tag{3}$$

and in the case of the plane stress state

$$\gamma = -\frac{\alpha E}{4\pi} R\int_0^{2\pi} f(\theta)\, e^{i\theta} d\theta. \tag{4}$$

Only this latter case will be examined below.

In the case of real γ (in principle, this can always be achieved by changing the order of reading θ), we obtain the following simple formulas for the stresses:

$$\left.\begin{aligned} \sigma_r &= \frac{2\gamma}{r}\left(1 - \frac{R^2}{r^2}\right)\cos\theta, \\ \sigma_\theta &= \frac{2\gamma}{r}\left(1 + \frac{R^2}{r^2}\right)\cos\theta, \end{aligned}\right\} \tag{5}$$

[1] Muskhelishvili, N.I. Nekotoryye osnovnyye zadachi matematicheskoy teorii uprugosti (Basic Problems of Mathematical Elasticity Theory). Moscow, Izdatel'stvo AN SSSR, 1954.

$$\left.\begin{aligned}\tau_{r\theta} &= \frac{2\gamma}{r}\left(1-\frac{R^2}{r^2}\right)\sin\theta,\\ \tau_{rz} &= \tau_{\theta z} = 0.\end{aligned}\right\} \tag{5}$$

At infinity, the stresses strive to zero, and the order of decrease is r^{-1}. Figure 1 presents graphs showing the dimensionless stresses $\frac{R}{2\gamma}\sigma_r$, $\frac{R}{2\gamma}\sigma\theta$ in the case $\theta = 0$ as a function of the dimensionless radius r/R. The solution obtained may be generalized to the case of elliptical and other types of holes for which we know the function which performs conformal mapping of the plane with a hole onto a plane with circular hole.

We shall assume that

$$f(\theta) = f_0 + A\cos\theta + \ldots, \tag{6}$$

and that there is no component with $\sin\theta$. Then

$$\gamma = -\frac{\alpha E}{4} RA - \tag{7}$$

is a real constant. Employing formula (1) from

$$\sigma_r + \sigma_\theta = 2\left[\Phi(z) + \overline{\Phi(z)}\right]$$

we may readily obtain ($\sigma_r = \tau_{r\theta} = 0$ for $r = R$) the expression σ_θ on the hole profile

$$\sigma_\theta = -\alpha EA\cos\theta \text{ for } r = R. \tag{8}$$

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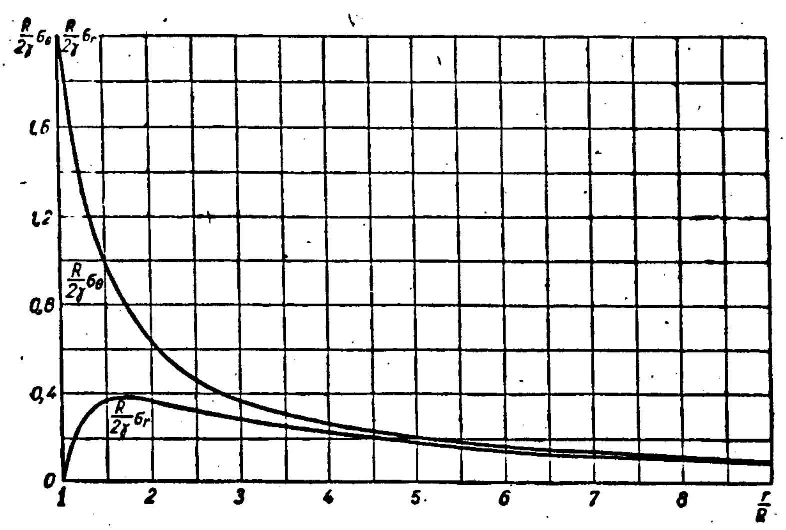

Figure 1

Let us apply the stress field corresponding to the Kirsch problem (for stresses at infinity $\sigma_x = p$, $\sigma_y = 0$, $\tau_{xy} = 0$) to the solution obtained. For the total problem, we obtain σ_θ on the hole profile in the following form

$$\sigma_\theta = p\left(3 - 4\cos^2\theta\right) - \alpha EA\cos\theta \text{ for } r = R. \tag{9}$$

The stress σ_x on the hole profile may be expressed by the following formula

$$\sigma_x = \sigma_\theta \sin^2\theta = [p(3-4\cos^2\theta) - \alpha E A \cos\theta]\sin^2\theta. \quad (10)$$

Let us introduce the stress concentration coefficient on the hole profile as follows:

$$K = \frac{\sigma_x}{p}; \quad (11)$$

we shall investigate the following coefficient along with it

$$k = \frac{\sigma_\theta}{p}. \quad (12)$$

If we introduce the notation

$$\frac{\alpha E A}{p} = a; \quad (13)$$

$$\cos\theta = u, \quad (14)$$

the coefficients K and k may be expressed as follows:

$$K(u) = (3 - 4u^2 - au)(1 - u^2); \quad (15)$$

$$k(u) = 3 - 4u^2 - au. \quad (16)$$

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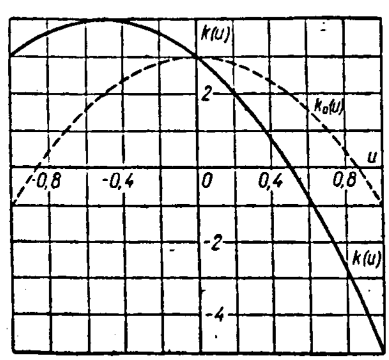

Figure 2

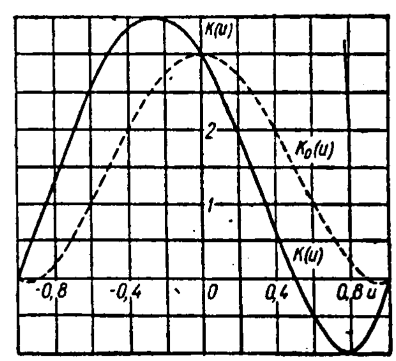

Figure 3

The stresses may be arranged symmetrically with respect to $\theta = 0$, and the quantity u changes between -1 and 1.

This is valid, if there is no plastic flow at any point in the region. This leads to the following inequality on the hole profile

$$-\tau_s\sqrt{3} < p(3-4u^2) - \alpha E A u < \tau_s\sqrt{3}, \quad (17)$$

and the yield condition of Mises is thus assumed. Under the Saint Venant-Trask condition, we must substitute $2\tau_s$ in (17), instead of $\tau_s\sqrt{3}$.

Example. Let the values of the main parameters be as follows:

$$\alpha = 12\cdot 10^{-6}\frac{1}{\text{deg.}};\ E = 2\cdot 10^6\frac{\text{dyne}}{\text{cm}^2};\ \tau_s = 2000\frac{\text{dyne}}{\text{cm}^2};\ p = 600\frac{\text{dyne}}{\text{cm}^2},$$

and the thermal field changes on the profile according to the following law

$$t(\theta) = t_0 + 100\cos\theta\ (^\circ\text{C}).$$

Then A = 100, a = 4,

$$k(u) = 3 - 4u^2 - 4u; \; K(u) = (1 - u^2)\, k(u).$$

The condition that there be no plastic flow on the profile is thus satisfied. Figure 2 and 3 present graphs showing the change in the coefficients k(u), K(u). The maximum k(u) is achieved in the case u = -0.5 (θ = 120°), and equals 4. The minimum is achieved in the case u = 1 (θ = 0) and equals -5. The maximum point K(u) is u = -0.250 (θ = 104°30'), and the maximum itself equals 3.5156. The minimum point is u = 0.781 (θ = 38°40'), and it equals -1.0000. For purposes of comparison, graphs of $k_0(u)$, $K_0(u)$ corresponding to the absence of a thermal gradient (a = 0) are presented.

We should note that in the general case, when the stress field at infinity has the following form

$$\sigma_x = p; \; \sigma_y = \delta p\, (\delta \neq 0); \; \tau_{xy} = 0,$$

it is advantageous to introduce two concentration coefficients characterizing stresses in the profile zone:

$$K_x = \frac{\sigma_x}{p}; \; K_y = \frac{\sigma_y}{\delta p}. \tag{18}$$

If we investigate the following coefficients /298

$$k_x = \frac{\sigma_\theta}{p}; \; k_y = \frac{\sigma_\theta}{\delta p} = \frac{k_x}{\delta}, \tag{19}$$

along with them, we then have

$$K_x = k_x \sin^2\theta, \; K_y = k_y \cos^2\theta = \frac{k_x}{\delta} \cos^2\theta. \tag{20}$$

Let the boundary function for temperature be expanded in Fourier series

$$f(\theta) = f_0 + A\cos\theta + B\sin\theta + \ldots, \tag{21}$$

and then

$$k_x(u) = 3 - 4u^2 + \delta(4u^2 - 1) - au - b\sqrt{1 - u^2}; \; k_y(u) = = \frac{1}{\delta} k_x(u), \tag{22}$$

$$K_x(u) = k_x(u)(1 - u^2); \; K_y(u) = \frac{1}{\delta} u^2 k_x(u), \tag{23}$$

where

$$b = \frac{\alpha E B}{p}. \tag{24}$$

The condition of no plastic flow on the profile may be expressed by the inequality

$$k(u) < \frac{\tau_s \sqrt{3}}{p}, \tag{25}$$

where τ_s is the yield limit in the case of pure shear.

EFFECT OF THE CREEP PROPERTIES OF THE MATERIAL ON THE STRESS CONCENTRATION AROUND A CIRCULAR HOLE IN A PLATE /299

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When the stress concentration around holes in plates and shells is investigated, it is interesting to study the influence of the inelastic behavior of the material upon the stress distribution around the holes. Studies in this area have only pertained to elastoplastic problems (Ref. 4). As of the present, there has been insufficient research devoted to the influence of the material creep properties upon the stress concentration, although it is of great practical importance for studying the supporting power of structural elements (Ref. 3).

This article studies the stress state of a plate with a circular hole on the basis of the nonlinear flow theory of steady creep. The problem may be solved by the method of successive approximations. The biaxial stress state of a plate and the particular cases following from it are investigated.

Formulation of the problem. We shall start with the stress-deformation relationships assumed in flow theory (Ref. 2)

$$\varepsilon_{ij}^{e}=\frac{1}{2\mu}\left(\sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij}\right)+\frac{1}{9K}\sigma_{kk}\delta_{ij},$$
$$\dot{\varepsilon}_{ij}^{p}=f(T)\left(\sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij}\right)\quad(i,\ j=1,\ 2,\ 3), \tag{1}$$

where ε_{ij}^{e} are elastic deformations; $\dot{\varepsilon}_{ij}^{p}$ -- rate of inelastic deformations; σ_{ij} -- stresses; μ -- shear modulus; K -- volume modulus; T -- second invariant of the stress tensor deviator,

$$T=\sigma_{11}^{2}+\sigma_{22}^{2}+\sigma_{33}^{2}-\sigma_{11}\sigma_{22}-\sigma_{11}\sigma_{33}-\sigma_{22}\sigma_{33}+3\left(\sigma_{12}^{2}+\sigma_{13}^{2}+\sigma_{23}^{2}\right). \tag{2}$$

A power law for the function f(T) is assumed in flow theory. In general, this law closely coincides with the experimental data, with the exception of stresses which are close to zero at which the rates of inelastic deformations are proportional to the stresses. It is usually assumed that this drawback is unimportant. However, in certain problems, it may qualitatively distort the actual process, as will be shown below. Therefore, we may write the form of the function f(T) as follows: /300

$$f(T)=a+bT^{n}. \tag{3}$$

In the case of the plane stress state, it follows from the relationships (1) that

$$\dot{\varepsilon}_{ij}=\frac{1}{2\mu}\left(\dot{\sigma}_{ij}-\frac{1}{3}\dot{\sigma}_{kk}\delta_{ij}\right)+\frac{1}{9K}\dot{\sigma}_{kk}\delta_{ij}+f(T)\left(\sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij}\right),$$
$$(i,\ j=1,\ 2), \tag{4}$$

where

$$\dot{\varepsilon}_{ij}=\dot{\varepsilon}_{ij}^{e}+\dot{\varepsilon}_{ij}^{p};\quad T=(\sigma_{11}+\sigma_{22})^{2}+3\left(\sigma_{12}^{2}-\sigma_{11}\sigma_{22}\right). \tag{5}$$

Employing the equation of compatability and the equation of equilibrium, and introducing the stress function F, we obtain the following equation for the stress function

$$\frac{\mu+3K}{9K\mu}\Delta\Delta\dot{F}+\frac{2}{3}a\Delta\Delta F=q. \tag{6}$$

The right hand portion q has the following form here

$$q=-b\left[\frac{2}{3}T^n\Delta\Delta F+\frac{\partial^2 T^n}{\partial x^2}\cdot\frac{\partial^2 F}{\partial x^2}+2\frac{\partial^2 T^n}{\partial x\,\partial y}\cdot\frac{\partial^2 F}{\partial x\,\partial y}+\frac{\partial^2 T^n}{\partial y^2}\cdot\frac{\partial^2 F}{\partial y^2}-\right.$$
$$\left.-\frac{1}{3}\Delta T^n\cdot\Delta F+\frac{4}{3}\left(\frac{\partial T^n}{\partial x}\cdot\frac{\partial\Delta F}{\partial x}+\frac{\partial T^n}{\partial y}\cdot\frac{\partial\Delta F}{\partial y}\right)\right]. \tag{7}$$

Equation (6) may be conveniently written as follows:

$$\Delta\Delta F=\frac{9K\mu}{\mu+3K}\int_0^t e^{-\frac{6aK\mu}{\mu+3K}(t-\tau)}q(\tau)\,d\tau+e^{-\frac{6aK\mu}{\mu+3K}t}\Delta\Delta F_0, \tag{8}$$

where F_0 is the stress function at the initial moment of time.

We will solve equation (8) by the method of successive approximations, assuming q = 0 in the first approximation, etc.

$$\Delta\Delta F^{(1)}=e^{-\frac{6aK\mu}{\mu+3K}t}\Delta\Delta F_0,$$
$$\Delta\Delta F^{(2)}=\frac{9K\mu}{\mu+3K}\int_0^t e^{-\frac{6aK\mu}{\mu+3K}(t-\tau)}q^{(1)}(\tau)\,d\tau+e^{-\frac{6aK\mu}{\mu+3K}t}\Delta\Delta F_0, \tag{9}$$
. . . .

Biaxial stress state of a plate with a circular hole. Let us investigate a plate with a circular hole under the condition that the hole profile is free, /301
and the following stresses are given at infinity

$$\sigma_x^\infty=p_1;\quad \sigma_y^\infty=p_2;\quad \tau_{xy}^\infty=0. \tag{10}$$

Thus, we may set n = 1 in the flow law (1), (3), which fully coincides with the experimental data for many materials. Introducing the dimensionless coordinate

$$\rho=\frac{r}{r_0}, \tag{11}$$

where r_0 is the hole radius, and assuming that the initial stress state is biharmonic ($\Delta\Delta F_0$ = 0), we may write equation (8) in the following form

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\cdot\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\cdot\frac{\partial^2}{\partial\theta^2}\right)F=$$
$$=\frac{9K\mu r_0^4}{\mu+3K}\int_0^t e^{-\frac{6aK\mu}{\mu+3K}(t-\tau)}q(\tau)\,d\tau, \tag{12}$$

where

$$q=-\frac{b}{r_0^3}\left\{\frac{2}{3r_0^2}T\Delta\Delta F+\frac{1}{3}\left[\sigma_\theta\left(2\frac{\partial^2 T}{\partial\rho^2}-\frac{1}{\rho}\cdot\frac{\partial T}{\partial\rho}-\frac{1}{\rho^2}\cdot\frac{\partial^2 T}{\partial\theta^2}\right)-\right.\right.$$
$$-\sigma_r\left(\frac{\partial^2 T}{\partial\rho^2}-\frac{2}{\rho}\cdot\frac{\partial T}{\partial\rho}-\frac{2}{\rho^2}\cdot\frac{\partial^2 T}{\partial\theta^2}\right)+4\left(\frac{\partial T}{\partial\rho}\cdot\frac{\partial(\sigma_r+\sigma_\theta)}{\partial\rho}+\frac{1}{\rho^2}\cdot\frac{\partial T}{\partial\theta}\times\right.$$
$$\left.\left.\left.\times\frac{\partial(\sigma_r+\sigma_\theta)}{\partial\theta}\right)\right]-2\tau_{r\theta}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\cdot\frac{\partial T}{\partial\theta}\right)\right\}. \tag{13}$$

Under the given boundary conditions, the first approximation of equation (12) has the following form, as is known (Ref. 4)

$$F^{(1)} = \frac{r_0^2}{2}(p_1 + p_2)\left(\frac{\rho^2}{2} - \ln\rho\right) + \frac{r_0^2}{2}(p_1 - p_2)\left(1 - \frac{\rho^2}{2} - \frac{1}{2\rho^2}\right)\cos 2\theta. \tag{14}$$

Determining the stresses and substituting them in (13), we obtain the equation for the second approximation

$$\Delta\Delta F^{(2)} = \frac{9K\mu r_0^4}{\mu + 3K}\int_0^t e^{-\frac{6aK\mu}{\mu+3K}(t-\tau)} q^{(1)}(\tau)\,d\tau, \tag{15}$$

where

$$\begin{aligned} q^{(1)} = -\frac{b}{r_0^2}\Big\{&(p_1 + p_2)^3\left(\frac{2}{\rho^6} + \frac{9}{\rho^8}\right) + (p_1 + p_2)(p_1 - p_2)^2\left(\frac{78}{\rho^8} - \frac{312}{\rho^{10}} + \right.\\ &\left. + \frac{540}{\rho^{12}}\right) + (p_1 + p_2)^2(p_1 - p_2)\left(-\frac{18}{\rho^6} + \frac{16}{\rho^8} - \frac{144}{\rho^{10}}\right)\cos 2\theta + \\ &+ (p_1 - p_2)^3\left(-\frac{72}{\rho^6} + \frac{408}{\rho^8} - \frac{732}{\rho^{10}} + \frac{756}{\rho^{12}} - \frac{810}{\rho^{14}}\right)\cos 2\theta + \\ &+ (p_1 + p_2)(p_1 - p_2)^2\left(\frac{21}{\rho^4} - \frac{72}{\rho^6} + \frac{180}{\rho^8} + \frac{81}{\rho^{12}}\right)\cos 4\theta + \\ &+ (p_1 - p_2)^3\left(\frac{36}{\rho^4} - \frac{122}{\rho^6}\right)\cos 6\theta\Big\}. \end{aligned} \tag{16}$$

The solution of equation (15) is as follows: /302

$$\begin{aligned} F^{(2)} = F^{(1)} - \frac{9K\mu b r_0^2}{8(\mu + 3K)}\int_0^t e^{-\frac{6aK\mu}{\mu+3K}(t-\tau)}\Big\{&(p_1 + p_2)^3\left(\ln\rho + \frac{1}{4\rho^2} + \frac{1}{8\rho^4}\right) + \\ &+ (p_1 + p_2)(p_1 - p_2)\left(\frac{97\ln\rho}{30} + \frac{13}{12\rho^4}\,\frac{13}{12\rho^6} + \frac{27}{40\rho^8}\right) + \Big[(p_1 + p_2)^2 \times \\ &\times (p_1 - p_2)\left(-\frac{71}{30} + \frac{79}{30\rho^2} + \frac{3\ln\rho}{\rho^2} + \frac{1}{3\rho^4} - \frac{3}{5\rho^6}\right) + (p_1 - p_2)^3 \times \\ &\times\left(-\frac{333}{140} - \frac{3057}{840\rho^2} + \frac{12\ln\rho}{\rho^2} + \frac{17}{2\rho^4} - \frac{61}{20\rho^6} + \frac{21}{20\rho^8} - \frac{27}{56\rho^{10}}\right)\Big]\cos 2\theta + \\ &+ (p_1 + p_2)(p_1 - p_2)^2\left(\frac{7}{8} - \frac{6\ln\rho}{\rho^2} + \frac{85}{14\rho^2} - \frac{9\ln\rho}{\rho^4} - \frac{199}{28\rho^4} + \right.\\ &\left. + \frac{9}{56\rho^6}\right)\cos 4\theta + (p_1 - p_2)^3\left(\frac{1}{4} - \frac{61}{40\rho^2} + \frac{23}{10\rho^4} - \frac{41}{40\rho^6}\right)\cos 6\theta\Big\}\,d\tau, \end{aligned} \tag{17}$$

where the loads p_1 and p_2 are functions of time. We obtain the following stresses from (17)

$$\begin{aligned} \sigma_r^{(2)} = \sigma_r^{(1)} - \frac{9K\mu b}{8(\mu + 3K)}\int_0^t e^{-\frac{6aK\mu}{\mu+3K}(t-\tau)}\Big\{&(p_1 + p_2)^3\left(\frac{1}{\rho^2} - \frac{1}{2\rho^4} - \frac{1}{2\rho^6}\right) + \\ &+ (p_1 + p_2)(p_1 - p_2)^2\left(\frac{97}{30\rho^2} - \frac{13}{3\rho^6} + \frac{13}{2\rho^8} - \frac{27}{5\rho^{10}}\right) + \Big[(p_1 + p_2)^2(p_1 - \\ &- p_2)\left(\frac{142}{15\rho^2} - \frac{192}{15\rho^4} - \frac{18\ln\rho}{\rho^4} - \frac{8}{3\rho^6} + \frac{6}{\rho^8}\right) + (p_1 - p_2)^3\left(\frac{333}{35\rho^2} + \frac{4737}{140\rho^4} - \right.\\ &\left. - \frac{72\ln\rho}{\rho^4} - \frac{68}{\rho^6} + \frac{61}{2\rho^8} - \frac{63}{5\rho^{10}} + \frac{27}{4\rho^{12}}\right)\Big]\cos 2\theta + (p_1 + p_2)(p_1 - p_2)^2 \times \\ &\times\left(-\frac{14}{\rho^2} - \frac{807}{\rho^4} + \frac{108\ln\rho}{\rho^4} + \frac{932}{7\rho^6} + \frac{180\ln\rho}{\rho^6} - \frac{27}{7\rho^{10}}\right)\cos 4\theta + (p_1 - p_2)^3 \times \\ &\times\left(-\frac{9}{\rho^2} + \frac{1159}{20\rho^4} - \frac{92}{\rho^6} + \frac{861}{20\rho^8}\right)\cos 6\theta\Big\}\,d\tau; \end{aligned} \tag{18}$$

$$\tau_{r\theta}^{(2)} = \tau_{r\theta}^{(1)} - \frac{9K\mu b}{8(\mu+3K)}\int_0^t e^{-\frac{6aK\mu}{\mu+3K}(t-\tau)}\left\{\left[(p_1+p_2)^2(p_1-p_2)\left(\frac{71}{15\rho^2} - \frac{49}{5\rho^4} - \frac{18\ln\rho}{\rho^4} - \frac{10}{3\rho^6} + \frac{42}{5\rho^8}\right) + (p_1-p_2)^3\left(\frac{333}{70\rho^2} + \frac{6417}{140\rho^4} - \frac{72\ln\rho}{\rho^4} - \frac{85}{\rho^6} + \frac{427}{10\rho^8} - \frac{189}{10\rho^{10}} + \frac{297}{28\rho^{12}}\right)\right]\sin 2\theta + (p_1+p_2)(p_1-p_2)^2\left(-\frac{7}{2\rho^2} + \frac{72\ln\rho}{\rho^4} - \frac{678}{7\rho^4} + \frac{743}{7\rho^6} + \frac{180\ln\rho}{\rho^6} - \frac{81}{14\rho^{10}}\right)\sin 4\theta + (p_1-p_2)^3\left(-\frac{3}{2\rho^2} + \frac{549}{20\rho^4} - \frac{69}{\rho^6} + \frac{861}{20\rho^8}\right)\sin 6\theta\right\}d\tau;$$

$$\sigma_\theta^{(2)} = \sigma_\theta^{(1)} - \frac{9K\mu b}{8(\mu+3K)}\int_0^t e^{-\frac{6aK\mu}{\mu+3K}(t-\tau)}\left\{(p_1+p_2)^3\left(-\frac{1}{\rho^2} + \frac{3}{2\rho^4} + \frac{5}{2\rho^6}\right) + (p_1+p_2)(p_1-p_2)^2\left(-\frac{97}{30\rho^2} + \frac{65}{3\rho^6} - \frac{91}{2\rho^8} + \frac{243}{5\rho^{10}}\right) + \left[(p_1+p_2)^2 \times (p_1-p_2)\left(\frac{4}{5\rho^4} + \frac{18\ln\rho}{\rho^4} + \frac{20}{3\rho^6} - \frac{126}{5\rho^8}\right) + (p_1-p_2)^3\left(-\frac{11457}{140\rho^4} + \frac{72\ln\rho}{\rho^4} + \frac{170}{\rho^6} - \frac{1281}{10\rho^8} + \frac{378}{5\rho^{10}} - \frac{1485}{28\rho^{12}}\right)\right]\cos 2\theta + (p_1+p_2)(p_1-p_2)^2 \times \left(\frac{765}{7\rho^4} - \frac{36\ln\rho}{\rho^4} - \frac{428}{7\rho^6} - \frac{180\ln\rho}{\rho^6} + \frac{81}{7\rho^{10}}\right)\cos 4\theta + (p_1-p_2)^3 \times \left(-\frac{183}{20\rho^4} + \frac{46}{\rho^6} - \frac{861}{20\rho^8}\right)\cos 6\theta\right\}d\tau. \tag{18}$$

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We may obtain the following particular cases from (18): unidirectional extension ($p_1 = p_2$), uniaxial extension ($p_2 = 0$), pure shear ($p_1 = -p_2$). We have the following stress on the hole profile

$$\sigma_\theta^{(2)} = (p_1+p_2) - 2(p_1-p_2)\cos 2\theta - \frac{9K\mu b}{8(\mu+3K)}\int_0^t e^{-\frac{6aK\mu}{\mu+3K}(t-\tau)}\left\{3(p_1+p_2)^3 + \frac{323}{15}(p_1+p_2)(p_1-p_2)^2 + \left[-\frac{266}{15}(p_1+p_2)^2(p_1-p_2) - \frac{608}{35}(p_1-p_2)^3\right]\cos 2\theta + \frac{118}{7}(p_1+p_2)(p_1-p_2)^2\cos 4\theta - \frac{31}{5}(p_1-p_2)^3\cos 6\theta\right\}d\tau. \tag{19}$$

In the case of uniaxial extension, we shall have

$$\sigma_\theta^{(2)} = p_1 - 2p_1\cos 2\theta - \frac{9K\mu b}{840(\mu+3K)}\int_0^t e^{-\frac{6aK\mu}{\mu+3K}(t-\tau)}p_1^3(\tau)\,d\tau\,(2576 - 3686\cos 2\theta + 1770\cos 4\theta - 651\cos 6\theta), \tag{20}$$

i.e., $\left(\theta = \pm\frac{\pi}{2}\right)$ at the most dangerous points

$$\sigma_\theta^{(2)} = 3p_1 - \frac{9K\mu b}{\mu+3K}\cdot\frac{8683}{840}\int_0^t e^{-\frac{6aK\mu}{\mu+3K}(t-\tau)}p_1^3(\tau)\,d\tau. \tag{21}$$

If we assume that the load p_1 is applied quite rapidly and remains constant, it follows from (19) that

$$\sigma_\theta^{(2)} = 3p_1\left[1 - \frac{8683b}{1680a}\left(1 - e^{-\frac{6aK\mu}{\mu+3K}t}\right)p_1^2\right], \tag{22}$$

i.e., the stress state is established with the passage of time, and the concentration coefficient k strives to the following value /304

$$k = 3\left(1 - \frac{8683b}{1680a} p_1^2\right). \tag{23}$$

The situation is entirely different in qualitative terms in the case a = 0. In this case, it follows from (22) that

$$\sigma_\theta^{(2)} = 3p_1\left[1 - \frac{8683K\mu b}{280(\mu + 3K)} t p_1^3\right]. \tag{24}$$

It may be readily seen that the results (17), (18) may be easily transferred to the case of biaxial stress state of a nonlinearly elastic plate with a hole (Ref. 1). For this purpose, we need only replace the time operators by the corresponding constants.

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PHYSICALLY NONLINEAR ELASTIC PLATES WEAKENED BY AN ARBITRARY HOLE /305

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Non-ferrous metals and their alloys, high-strength steels, rubbers, polymers, and glass-like plastics are used more and more in modern technology. Even for small deformations, these materials do not obey Hooke's law, and therefore it is necessary to take into account the nonlinear dependence of deformation on stress. Nonlinearity of this type is called physical nonlinearity.

The influence of physical nonlinearity upon the stress state in the stress concentration zone in a plate with a hole is of particular interest. The stresses are several times greater in the concentration zone than they are outside of this zone. Therefore, the relationship between stresses and deformations outside the zone will be linear, and it will be a nonlinear relationship in the concentraion zone for one and the same external loading.

This article investigates the stress state around a square hole with reinforced corners, in a plate, for nonlinear dependence between stresses and deformations, which was advanced in (Ref. 2).

Since the nonlinearly deformed material under consideration is an ideally elastic material, the objects being studied have an elastic potential. Representing the potential as the function of the deformation invariants

$$A = A(J_1, J_2, J_3), \tag{1}$$

where J_1, J_2, J_3 are the deformation tensor invariants, and taking the fact into account that the volumetric deformation follows Hooke's law in the region of small deformations, we obtain a nonlinear elasticity law for the generalized /306 plane stress state in the following form

$$\left.\begin{aligned} e_r &= \frac{1}{3K}\sigma_0 + \frac{1}{2G}(\sigma_r - \sigma_0) + \frac{g_2}{2G}t_0^2(\sigma_r - \sigma_0);\\ e_\varphi &= \frac{1}{3K}\sigma_0 + \frac{1}{2G}(\sigma_\varphi - \sigma_0) + \frac{g_2}{2G}t_0^2(\sigma_\varphi - \sigma_0);\\ e_{r\varphi} &= \frac{1}{G}\tau_{r\varphi} + \frac{g_2}{G}t_0^2\tau_{r\varphi}, \end{aligned}\right\} \tag{2}$$

where K and G are, respectively, the modulus of volumetric deformation and the shear modulus; e_r, e_ϕ, $e_{r\phi}$ and σ_r, σ_ϕ, $\tau_{r\phi}$ -- the mean values (with respect to the plate thickness) of the components of deformation and stress, respectively, in a polar coordinate system (r, ϕ); the mean stress σ_0 and the intensity of the shearing stresses t_0^2 have the following form

$$\sigma_0 = \frac{1}{3}(\sigma_r + \sigma_\varphi);\quad t_0^2 = \frac{\tau_0^2}{G^2} = \frac{2}{9G^2}(\sigma_r^2 + \sigma_\varphi^2 - \sigma_r\sigma_\varphi + 3\tau_{r\varphi}^2). \tag{3}$$

The dimensionless constant g_2 for certain materials (Ref. 6) has the order 10^5 - 10^6, and may be determined according to the following formula

$$g_2 = 4{,}5G^2\left(1 + \frac{G}{3K}\right)a_2, \tag{4}$$

where the parameter α_3 is determined from the σ - e_{xx} diagram for uniaxial extension of a sample made of a material which is physically nonlinear.

When the elasticity law (2) is selected, the solution of the plane physically nonlinear problem may be reduced to integrating the nonlinear equation of the following type (Ref. 7)

$$\Delta\Delta F - \lambda L\left(F, \frac{\partial F}{\partial r}, \frac{\partial F}{\partial \varphi}, \frac{\partial^2 F}{\partial r^2}, \frac{\partial^2 F}{\partial \varphi^2}, \frac{\partial^2 F}{\partial r \partial \varphi}\right) = 0. \qquad (5)$$

The operator L depends nonlinearly upon the stress function and its derivatives; $\Delta = F_{rr} + \frac{F_r}{r} + \frac{1}{r^2} F_{\phi\phi}$ -- Laplace operator[1]; the parameter λ designates the deviation of the nonlinear elasticity law from Hooke's law, has the dimensionality bar^{-2}, a magnitude on the order of 10^{-5} - 10^{-6}, and is determined according to the following formula

$$\lambda = \frac{K}{(3K+G)} \cdot \frac{g_2}{G^2} = \mu\beta^2; \quad \mu = \frac{1}{g_2}; \quad \beta^2 = \frac{K}{3K+G} \cdot \frac{g_2^2}{G^2}. \qquad (6)$$

Let us study the stress state of an unbounded, physically nonlinear isotropic plate with a square hole under unidirectional, uniform tension by stresses p at infinity.

The study (Ref. 1) advanced an approximate method for solving the problem /307 of stress concentration around a curvilinear hole in a plate for the elasticity law given above. This method consists of the fact that, for any hole obtained from the mapping function

$$\omega(\zeta) = R[\zeta + \varepsilon f(\zeta)], \qquad (7)$$

the solution of the problem under consideration may be obtained by expanding the unknown stress function $F(r, \phi)$ (5) in double series with respect to the small parameters μ and ε (one of them characterizes the physical nonlinearity of the material, and the other characterizes the curvilinearity of the hole)

$$F(r; \varphi; \mu; \varepsilon) = H_0 \sum_{i,j=0}^{\infty} \mu^i \varepsilon^j F^{(i,j)}(r, \varphi). \qquad (8)$$

In formulas (7), (8) $\varepsilon < 1$; R -- real constant characterizing the hole dimensions; $f(\zeta)$ is selected from the condition $H_0\beta$: $R^2 = 1$; $\zeta = \rho e^{i\theta}$.

We may obtain the solution by retaining the following number of terms in the series (8)

$$F(r; \varphi; \mu; \varepsilon) = H_0[F^{(0,0)}(r, \varphi) + \mu F^{(1,0)}(r, \varphi) + \mu^2 F^{(2,0)}(r, \varphi) + \varepsilon F^{(0,1)}(r, \varphi) + \varepsilon^2 F^{(0,2)}(r, \varphi) + \mu\varepsilon F^{(1,1)}(r, \varphi)]. \qquad (9)$$

In expression (9) the functions $F^{(i,j)}(r, \phi)$ ($i = 0; 1; 2$; $j = 0; 1; 2$) correspond to: $F^{(0,0)}(r, \phi)$ -- the known solution of the linear problem for stress concentration in a plate with a circular hole (Ref. 3)

$$F^{(0,0)} = \frac{p}{2H_0}(r^2 - 2\ln r); \qquad (10)$$

where $F^{(1,0)}(r, \phi)$ is the solution of the same problem obtained by taking into

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[1] The partial derivatives of F with respect to r and ϕ are designated by the corresponding subscripts.

account the physical nonlinearity in the first approximation (Ref. 6)

$$F^{(1,0)} = -\frac{p^3}{H_0^3}\left[\frac{1}{4}\left(r^{-2}+\frac{1}{2}r^{-4}\right)+\ln r\right]; \tag{11}$$

$F^{(2,0)}(r, \phi)$ is the solution of the same problem obtained by taking into account the physical nonlinearity in the second approximation (Ref. 5)

$$F^{(2,0)} = \frac{p^5}{H_0^5}\left[\frac{13}{5}\ln r+\frac{1}{4}\left(-r^{-2}-\frac{1}{3}r^{-4}+\frac{47}{36}+r^{-6}+\frac{59}{80}r^{-8}\right)\right]; \tag{12}$$

$F^{(0,1)}(r, \phi)$ and $F^{(0,2)}(r, \phi)$ is the solution obtained for the linear problem for unidirectional tension of an infinite plate with a square hole (Ref. 4) taking ε and ε^2 into account,

$$\begin{gathered} F^{(0,1)}(r,\varphi) = -\frac{p}{H_0}(r^{-2}-r^{-4})\cos 4\varphi, \quad F^{(0,2)}(r,\varphi) = \\ = \frac{p}{H_0}\left[\left(\frac{7}{2}r^{-8}-3r^{-6}\right)\cos\varphi - 3\ln r\right]. \end{gathered} \tag{13}$$

For the case under consideration, the function (7) may be selected in the following form /308

$$\omega(\zeta) = \left(\zeta+\frac{1}{9}\zeta^{-3}\right), \tag{14}$$

where $F^{(1,1)}(r, \phi)$ is the desired solution for a physical nonlinear elastic plate with a square hole. We may obtain the equation for the function $F^{(1,1)}$ by substituting the function (8) in equation (5) and equating the coefficients in the left and right hand sides in the case $\mu\varepsilon$

$$\begin{gathered} \Delta\Delta F^{(1,1)} = T^{(0,0)}_{r,r}(r^{-2}F^{(0,1)}_{\varphi,\varphi}+r^{-1}F^{(0,1)}_{r}) + T^{(0,1)}_{r,r}(r^{-2}F^{(0,0)}_{\varphi,\varphi}+r^{-1}F^{(0,0)}_{r}) + \\ + F^{(0,1)}_{r,r}(r^{-2}T^{(0,0)}_{\varphi,\varphi}+r^{-1}T^{(0,0)}_{r}) + F^{(0,0)}_{r,r}(r^{-2}T^{(0,1)}_{\varphi,\varphi}+r^{-1}T^{(0,1)}_{r}) - \\ - 2\,[(r^{-1}T^{(0,0)}_{r,\varphi}-r^{-2}T^{(0,0)}_{\varphi})(r^{-1}F^{(0,1)}_{r,\varphi}-r^{-2}F^{(0,1)}_{\varphi}) + (r^{-1}F^{(0,0)}_{r,\varphi} - \\ - r^{-2}F^{(0,0)}_{\varphi})(r^{-1}T^{(0,1)}_{r,\varphi}-r^{-2}T^{(0,1)}_{\varphi})] - \frac{2}{3}\Delta\,(T^{(0,1)}\Delta F^{(0,0)}+T^{(0,0)}\Delta F^{(0,1)}). \end{gathered} \tag{15}$$

The functions $T^{(0,0)}(r, \phi)$ and $T^{(0,1)}(r, \phi)$ are given in the study (Ref. 1).

In order to find the stress components σ_ρ, σ_θ, $\tau_{\rho\theta}$ in the curvilinear[1] orthogonal coordinate system (ρ, θ) we must employ (Ref. 4) the formulas for changing from polar coordinates (r, ϕ) to the coordinates (ρ, θ). The stress components σ_ρ, σ_θ, $\tau_\rho\theta$, just as in (Ref. 8), may be represented in the form of a double series with respect to μ and ε. The stress components corresponding to the functions (9) $F^{(i,j)}(r, \phi)$ may be determined according to the formulas

$$\left.\begin{gathered} \sigma_\theta^{(0,0)} = H_0\frac{\partial^2}{\partial\rho^2}F^{(0,0)}_{(\rho,\theta)};\ \sigma_\theta^{(0,1)} = H_0\frac{\partial^2}{\partial\rho^2}F^{(1,0)}_{(\rho,\theta)};\ \sigma_\theta^{(0,2)} = H_0\frac{\partial^2}{\partial\rho^2}F^{(2,0)}_{(\rho,\theta)}; \\ \sigma_\theta^{(0,1)} = H_0\frac{\partial^2}{\partial\rho^2}F^{(0,1)}_{(\rho,\theta)} + H_0L_1^{(1)}\frac{\partial^2}{\partial\rho^2}F^{(0,0)};\ \sigma_\theta^{(0,2)} = H_0\frac{\partial^2}{\partial\rho^2}F^{(0,2)}_{(\rho,\theta)} + \\ + H_0\left[L_1^{(2)}\frac{\partial^2}{\partial\rho^2}+L_2^{(2)}\left(\Delta-2\frac{\partial^2}{\partial\rho^2}\right)+L_3^{(2)}\frac{\partial^2}{\partial\rho\,\partial\theta}\cdot\frac{1}{\rho}\right]F^{(0,0)} + \\ + H_0\left[L_1^{(1)}\frac{\partial^2}{\partial\rho^2}+L_2^{(1)}\left(\Delta-2\frac{\partial^2}{\partial\rho^2}\right)+L_3^{(1)}\frac{\partial^2}{\partial\rho\,\partial\theta}\cdot\frac{1}{\rho}\right]F^{(0,1)}; \end{gathered}\right\} \tag{16}$$

[1] The coordinate line $\rho = 1$ coincides with the hole profile.

$$\sigma_\theta^{(1,1)} = H_0 \frac{\partial^2}{\partial\rho^2} F^{(1,1)}_{(\rho,\theta)} + H_0 \Big[L_1^{(1)} \frac{\partial^2}{\partial\rho^2} + L_2^{(1)} \Big(\Delta - 2\frac{\partial^2}{\partial\rho^2}\Big) + \\ + L_3^{(1)} \frac{\partial^2}{\partial\rho\,\partial\theta} \cdot \frac{1}{\rho} \Big] F^{(1,0)}_{(\rho,\theta)}. \quad (16)$$

The operators L_i^j depend on the function (14) and may be determined according to the formulas

$$L_1^{(1)} = \rho^{-3}\cos 4\theta \frac{\partial}{\partial\rho} - \rho^{-4}\sin 4\theta \frac{\partial}{\partial\theta}; \quad L_1^{(2)} = \frac{1}{4}\rho^{-6}(1+\cos 8\theta)\frac{\partial^2}{\partial\rho^2} - \\ - \frac{1}{2}\rho^{-6}\sin 8\theta \frac{\partial^2}{\partial\rho\,\partial\theta}\frac{1}{\rho} + \frac{1}{4}\rho^{-8}(1-\cos 8\theta)\Big(\frac{\partial^2}{\partial\theta^2} + \rho\frac{\partial}{\partial\rho}\Big); \\ L_2^{(2)} = 8\rho^{-8}(1-\cos 8\theta); \quad L_3^{(2)} = 8\rho^{-8}\cos 8\theta + 4\rho^{-7}\sin 8\theta \frac{\partial}{\partial\rho} - \\ - 4\rho^{-8}(1-\cos 8\theta)\frac{\partial}{\partial\theta}; \quad L_2^{(1)} = 0; \quad L_3^{(1)} = 8\rho^{-4}\sin 4\theta. \quad (17)$$

The functions $F^{(i,\, j)}(r, \phi)$ contained in (16) represent the solution for an equation like (15) in the form of Fourier series: /309

$$F^{(i,j)}(r, \varphi) = \sum_{k=0}^{\infty} [f_{ij}^{(k)}(r)\sin k\varphi + g_{ij}^{(k)}(r)\cos k\varphi], \quad (18)$$

in which the variables r and ϕ are replaced by ρ and θ, respectively.

Substituting the functions (10), (13) and their derivatives in equation (15), we obtain

$$\Delta\Delta F^{(1,1)} = 48\frac{p^2}{H_0^3}(56r^{-10} - 270r^{-12})\cos 4\varphi. \quad (19)$$

We may represent the solution of this equation in the form of the superposition of the particular integral of this equation

$$F^{(1,1)}_{\text{part.}}(r, \varphi) = \frac{p^2}{H_0^3}(2{,}800r^{-6} - 3{,}214r^{-8})\cos 4\varphi \quad (20)$$

and the integral of the homogeneous equation $\Delta\Delta F^{(1,\,1)} = 0$

$$F^{(1,1)}_{\text{homo}}(r, \varphi) = \sum_{n=2}^{\infty}(C_{n3}r^{-n+2} + C_{n4}r^{-n})\cos n\varphi. \quad (21)$$

We may determine the integration constants C_{n3} and C_{n4} from different conditions over the hole profile in the case $\rho = 1$. The final expression for the function $F^{(1,\,1)}$ will be

$$F^{(1,1)}(\rho, \theta) = -\frac{p^2}{H_0^3}(2{,}128\rho^{-2} - 2{,}543\rho^{-4} - 2{,}80\rho^{-6} + 3{,}214\rho^{-8})\cos 4\theta. \quad (22)$$

Taking into account the functions (10), (13) and (22), we may determine the stress state of a physically nonlinear elastic plate with a square hole (14) by means of formulas (16), (17), within the given degree of accuracy (9). Let us present the concentration coefficient values along the profile of a square hole

$$k = \Big(\frac{\sigma_\theta}{p}\Big)_{\rho=l} = 2(1 - 1{,}500\lambda p^2 + 10{,}608\lambda^2 p^4 + 0{,}666\cos 4\theta + \\ + 0{,}197\cos 8\theta - 3{,}152\lambda p^2\cos 4\theta). \quad (23)$$

Tables 1-4 present the values of the concentration coefficient (23) for linear and nonlinear theory over the hole profile [in view of the complete symmetry, the values are presented from 0-45° for a different magnitude of the external load p and different materials (Ref. 6, 8)].

Table 1 presents the values of the concentration coefficient for copper $\lambda = \lambda_1 = 1.019\cdot10^{-5}$ bar^{-2}; Table 2 presents the values for pure copper $\lambda = \lambda_2 = 0.266\cdot10^{-6}$ bar^{-2}. Table 3 presents the values for aluminum bronze $\lambda = \lambda_3 = 0.055\cdot10^{-6}$ bar^{-2}, and Table 4 presents the values for open-hearth steel $\lambda = \lambda_4 = 0.033\cdot10^{-6}$ bar^{-2}.

TABLE 1 /310

Theory	Load p, dyne	Angle θ: 0	5	10	20	35	45
Nonlinear	40	3,5799	3,4136	2,9668	1,7999	1,0835	1,1215
	50	3,5027	3,3399	2,9031	1,7705	1,1084	1,1599
	60	3,4132	3,2547	2,8301	1,7394	1,1437	1,2117
	70	3,3143	3,1608	2,7508	1,7095	1,1922	1,2799
	75	3,2624	3,1117	2,7097	1,6961	1,2226	1,3211
	80	3,2095	3,0617	2,6684	1,6842	1,2576	1,3677
	85	3,1560	3,0115	2,6274	1,6746	1,2977	1,4202
Linear	—	3,7260	3,5535	3,0888	1,8610	1,0480	1,0620

TABLE 2

Theory	Load p, dyne	Angle θ: 0	5	10	20	35	45
Nonlinear	100	3,7014	3,5299	3,0681	1,8503	1,0531	1,0709
	200	3,6294	3,4609	3,0079	1,8200	1,0699	1,0995
	300	3,5154	3,3520	2,9135	1,7752	1,1040	1,1532
	400	3,3684	3,2121	2,7940	1,7252	1,1643	1,2100
	450	3,2864	3,1343	2,7286	1,7021	1,2081	1,3015
	500	3,2011	3,0539	2,6220	1,6820	1,2635	1,3755
	550	3,1140	2,9728	2,5962	1,6689	1,3326	1,4652
	600	3,0296	2,8935	2,5336	1,6635	1,4177	1,5729
	650	2,9483	2,8185	2,4769	1,6688	1,5216	1,7013
Linear	—	3,7260	3,5535	3,0688	1,8610	1,0480	1,0620

We may see from these computations that the stress state in the zone of stress concentration is distributed more uniformly than in the linear case for the given variation of a physically nonlinear elasticity theory. The concentration coefficient depends essentially on the elastic properties of the material, the magnitude of the external load, and the nature of the mapping function. Comparing the concentration coefficient value determined from a precise solution (Ref. 3) and the approximate value (Ref. 4) for Hooke's law and for different approximations in the case of a nonlinear theory (Ref. 5), we obtain an idea of the rapidity of convergence of the method employed to solve physically nonlinear problems of stress concentration around arbitrary holes. /311

TABLE 3

Theory	Load p, dyne	Angle θ					
		0	5	10	20	35	45
Nonlinear	200	3,7056	3,5340	3,0718	1,8521	1,0521	1,0694
	300	3,6805	3,5098	3,0505	1,8413	1,0576	1,0789
	400	3,6458	3,4766	3,0215	1,8276	1,0658	1,0927
	500	3,6021	3,4348	2,9851	1,8088	1,0772	1,1114
	600	3,5501	3,3851	2,9421	1,7883	1,0926	1,1357
	700	3,4907	3,3284	2,8932	1,7661	1,1127	1,1664
	800	3,4248	3,2657	2,8395	1,7432	1,1387	1,2046
	900	3,3536	3,1980	2,7821	1,7207	1,1716	1,2513
	1000	3,2784	3,1269	2,7224	1,7000	1,2128	1,3079
	1200	3,1222	2,9798	2,6018	1,6698	1,3260	1,4567
Linear	—	3,7260	3,5535	3,0888	1,8610	1,0480	1,0620

TABLE 4

Theory	Load p, dyne	Angle θ					
		0	5	10	20	35	45
Nonlinear	500	3,6507	3,4813	3,0256	1,8287	1,0645	1,0907
	600	3,6185	3,4505	2,9988	1,8154	1,0728	1,1042
	700	3,5811	3,4147	2,9697	1,8004	1,0832	1,1210
	800	3,5390	3,3745	2,9329	1,7840	1,0961	1,1412
	900	3,4925	3,3301	2,8947	1,7668	1,1121	1,1655
	1000	3,4421	3,2821	2,8535	1,7490	1,1315	1,1941
	1100	3,3883	3,2310	2,8100	1,7314	1,1549	1,2277
	1200	3,3318	3,1773	2,7646	1,7144	1,1829	1,2669
Linear	—	3,7260	3,5335	3,0888	1,8610	1,0480	1,0620

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DETERMINATION OF THE STRESS CONCENTRATION NEAR A HOLE IN A SHELL - IN THE LINEAR FORMULATION /312

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N67-24542

The problem of determining the stress concentration around a hole in a shell having a general form was advanced in the study by G. N. Savin (Ref. 4). This problem is characterized by the following requirement: it is necessary to determine the concentration coefficient $K = \frac{\sigma_{max}}{\sigma_0}$ corresponding to the increase in stress due to the occurrence of a hole (σ_0 -- stress for the unperturbed state).

Problems on concentration are very time-consuming. Therefore, it is valid to present and discuss considerations which may facilitate the solution of this problem to a certain extent. This article is devoted to the discussion of some of these considerations.

It is apparent that the hole under consideration must not be very small, since, for a hole with a radius on the order of the thickness, a three-dimensional stress state occurs in the vicinity of the hole; this stress state is not described by the relationships of the theory of thin shells. There is not much interest in investigating small holes (punctures), since they are reinforced very simply in practice.

At the same time, the specific properties of the problem disappear for very large holes. Essentially, in these cases we are dealing with the calculation of a shell with an edge. Usually, such problems may be solved by dividing the desired solution into the main solution (which is most frequently the solution with zero moment) and the simple (exponential) edge effect.

It is possible and expedient to introduce a certain quantitative aspect into the concept "the "upper limit". For example, by employing the known solution of the stress concentration around a circular hole in a spherical shell we may assume that we are dealing with a problem of stress concentration /313 around a hole, and the customary (exponential) asymptotic behavior may not be employed. This criterion for the hole smallness [which depends on the thinness of the shell walls and the magnitude of the permissible error, see (Ref. 3)] can be speculatively transferred to arbitrary shells with holes having a general form.

The local nature of the problem (Ref. 4) makes it possible to employ the equations of shallow shells advanced by V. Z. Vlasov. In our opinion, it is more expedient to imply the somewhat modified equation of V.Z. Vlasov [(Ref. 6), page 182]

$$\Delta\Delta\tilde{w} + \frac{i}{c} D\tilde{w} = -\frac{q_n^*}{Ehc} - \frac{i}{c} q_n^u, \tag{1}$$

where we have the following in arbitrary orthogonal coordinates

$$
\begin{gathered}
\Delta = \frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\cdot\frac{B}{A}\cdot\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta}\cdot\frac{A}{B}\cdot\frac{\partial}{\partial\beta}\right\};\\
D = \frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\cdot\frac{B}{AR_\beta}\cdot\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta}\cdot\frac{1}{R_{\alpha\beta}}\cdot\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\alpha}\cdot\frac{1}{R_{\alpha\beta}}\cdot\frac{\partial}{\partial\beta}+\frac{\partial}{\partial\beta}\cdot\frac{A}{BR_\alpha}\cdot\frac{\partial}{\partial\beta}\right\};\\
q_n^* = \frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\cdot\frac{1}{A}\left(\frac{\partial BM_\alpha^*}{\partial\alpha}+\frac{1}{A}\cdot\frac{\partial A^2H^*}{\partial\beta}-\frac{\partial B}{\partial\alpha}M_\beta^*\right)+\frac{\partial}{\partial\beta}\cdot\frac{1}{B}\left(\frac{\partial AM_\beta^*}{\partial\beta}+\right.\right.\\
\left.\left.+\frac{1}{B}\cdot\frac{\partial B^2H^*}{\partial\alpha}-\frac{\partial A}{\partial\beta}M_\alpha^*\right)\right\};\\
q_n^u = \frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\cdot\frac{1}{A}\left(\frac{\partial B\varepsilon_\beta^u}{\partial\alpha}-\frac{1}{A}\cdot\frac{\partial A^2\frac{\omega^u}{2}}{\partial\beta}-\frac{\partial B}{\partial\alpha}\varepsilon_\alpha^u\right)+\right.\\
\left.+\frac{\partial}{\partial\beta}\cdot\frac{1}{B}\left(\frac{\partial A\varepsilon_\alpha^u}{\partial\beta}-\frac{1}{B}\cdot\frac{\partial B^2\frac{\omega^u}{2}}{\partial\alpha}-\frac{\partial A}{\partial\beta}\varepsilon_\beta^u\right)\right\}.
\end{gathered}
\qquad (2)
$$

Thus the complex stresses may be determined by the following relationships

$$
\begin{gathered}
\widetilde{T}_\alpha = T_\alpha^* - iEhc\varkappa_\beta^u +\\
+iEhc\left\{\frac{1}{B}\cdot\frac{\partial}{\partial\beta}\cdot\frac{1}{B}\cdot\frac{\partial\widetilde{w}}{\partial\beta}+\frac{1}{AB}\cdot\frac{\partial B}{\partial\alpha}\cdot\frac{1}{A}\cdot\frac{\partial\widetilde{w}}{\partial\alpha}\right\}_{\alpha\rightleftarrows\beta};\\
\widetilde{S} = S^* + iEhc\tau^u +\\
+iEhc\left\{-\frac{1}{AB}\left(\frac{\partial^2\widetilde{w}}{\partial\alpha\partial\beta}-\frac{1}{A}\cdot\frac{\partial A}{\partial\beta}\cdot\frac{\partial\widetilde{w}}{\partial\alpha}-\frac{1}{B}\cdot\frac{\partial B}{\partial\alpha}\cdot\frac{\partial\widetilde{w}}{\partial\beta}\right)\right\}.
\end{gathered}
$$

These relationships include a system of static functions $\{T_\alpha^*, T_\beta^*, \ldots, H^*\}$, which satisfies the equations of equilibrium, and a system of geometric functions $\{\varepsilon_\alpha^u, \varepsilon_\beta^u, \ldots, \tau^u\}$ which satisfies the equations of continuity of the middle surface. In other respects, the functions introduced are arbitrary.

The form which is advanced is more flexible than the customary form. In principle, it enables us "pick out" the main portion of the solution obtained when the static and geometric systems are successfully selected. This may be done by reducing the role of terms depending on $\widetilde{w}$ to corrections [(Ref. 6), page 183]. /314

In addition, in the case of loads which are not self-balancing over the hole profile, we may guarantee the single-valued nature of the stress functions obtained by appropriate selection of the functions for the static system.

It is very promising and convenient to employ the isothermal coordinates related to the conformal mapping [in our notation (Ref. 6), page 69] introduced by G. N. Savin for studying this problem:

$$
z = \alpha + i\beta = \omega(\zeta);\ \zeta = e^{\rho+i\psi}. \qquad (3)
$$

These coordinates make it possible to map the region with a hole onto the interior of the circle $\rho \leqq \rho_0$ (Figure 1).

The condition of a freely-supported cover is most general for the hole profile

$$
\begin{gathered}
Q_{\rho\rho} = Q_{\rho\psi} = 0;\quad Q_{\rho n} = Q_{\rho n}^0;\\
M_{\rho\rho} = 0,
\end{gathered}
\qquad (4)
$$

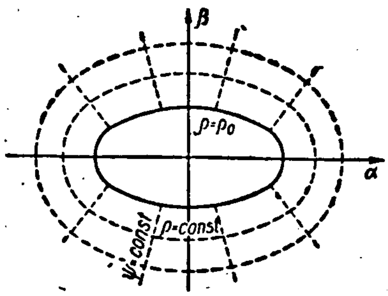

Figure 1

where [see, for example, (Ref. 4, 1)] for an elliptical hole with the semi-axes a and b

$$Q_{\rho n}=p\frac{b}{a+b}[a-(a-b)\cos 2\psi], \quad (5)$$

the third condition of (4) is replaced by the less restrictive requirement balancing of pressure on the cover by the intersection stress

$$\int\limits_{\rho=\rho_0} Q_{\rho n}(\vec{n}\cdot\vec{n}_0)\,ds_\psi = pS. \quad (6)$$

This is directed along the normal to the surface at the cover center $(\vec{n}_0)$; S is the hole area. In view of the relative smallness of the hole, it is usually assumed that $\vec{n}\cdot\vec{n}_0 = 1$ (Ref. 2).

In the studies employing an a priori law for the distribution of intersection stresses (5), or which stipulate that (6) is satisfied on the average, these assumptions represent the most vulnerable point. The problem of determining the stress concentration magnitude pertains to the immediate vicinity of the boundary profile, and the redistribution of the intersection stress may have a significant influence influence on the concentration coefficient. It is of value to make the following numerical experiment: let us compare the con- /315
centration coefficients for a solution based on condition (5), and for a solution employing a distribution which is uniform along the profile, for example.

The condition of a rigid cover is much less vulnerable (see Figure 2)

$$Q_{\rho\rho}=Q_{\rho\psi}=0;\quad w=w^0;\quad M_{\rho\rho}=0, \quad (7)$$

where $w^0 = w(A)f(\psi)$.

Employing the fact that the cover is rigid, let us set w(A) as the normal displacement of the cover center. Then, if we know the cover form, we may readily determine the function $f(\psi)$, i.e., we may define the law governing the distribution of normal displacement over the hole profile.

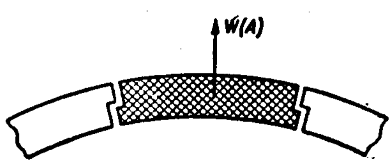

Figure 2

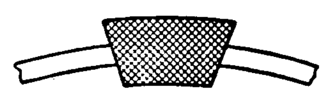

Figure 3

The parameter w(A) may be determined from the condition of the cover equilibrium. We may introduce the remaining five parameters determining the cover displacement as a rigid body, and we may investigate the case in which the influence of the general form is applied to the cover.

The boundary condition under consideration is usually approximately realized, since, as a rule, covers are more rigid than the edge of a shell. The case of a free edge is no less important:

$$Q_{\rho\rho} = Q_{\rho\psi} = Q_{\rho n} = 0; \quad M_{\rho\rho} = 0. \tag{8}$$

The calculation of this case differs from the preceding calculation in the fact that, if we remove (as is customarily done) the surface load by a zero-moment solution, for the homogeneous problem we obtain an edge loading which is non-selfbalancing on the profile. Due to this fact, the stress function $w = \mathrm{Im}\,\tilde{w}$ will be multivalued [(Ref. 6), page 156]. We may avoid this by employing the modified relationships (1) - (2) (as was already indicated).

We should point out that the conditions of the free edge are the most "suited" for confirming the proposed computational methods. Conditions (8) may be realized in experiments with a sufficient degree of accuracy by establishing corresponding tensions (Ref. 5).

The case of the rigid edge (Figure 3) is the most important case in applications. It may be formulated in the following form [(Ref. 6), page 121]

$$\varkappa_{\psi\psi} = \varkappa_{\psi\rho} = \varkappa_{\psi n} = 0; \quad \varepsilon_{\psi\psi} = 0\,[w = w^0 = w(A)\,f(\psi)]. \tag{9}$$

The first four of the conditions presented indicate that the edge is not deformed. The fourth condition (which does not contradict the first three) determines the translational motion of the cover. /316

In contrast to the freely geometric conditions (displacements and rotational angle are given), the conditions described may be formulated in terms of the basic complex function

$$\tilde{w} = w + i\bar{w}. \tag{10}$$

We should note that in essence the case under consideration includes the entire problem of designing shells which are loaded over a small section of the surface by concentrated forces, in particular.

Conditions of elastic coupling with a connecting piece (ring) are similar to the conditions actually existing in shells. They may be formulated as follows in terms of static and deformation boundary values:

$$\left.\begin{array}{c} \vec{Q}_\rho = \vec{Q}_\rho^{\,C}, \quad M_{\rho\rho} = M_{\rho\rho}^{C}; \\ \vec{\varkappa}_\psi = \vec{\varkappa}_\psi^{\,C}, \quad \varepsilon_{\psi\psi} = \varepsilon_{\psi\psi}^{C} \\ \left(\begin{array}{c} \vec{Q}_\rho = Q_{\rho\rho}\vec{e}_\rho + Q_{\rho\psi}\vec{e}_\psi + Q_{\rho n}\vec{c} \\ \vec{\varkappa}_\psi = \varkappa_{\psi\psi}\vec{e}_\rho + \varkappa_{\psi\rho}\vec{e}_\psi + \varkappa_{\psi n}\vec{n} \end{array}\right). \end{array}\right\} \tag{11}$$

If the edges of the shell and the connecting piece have the same thickness and are made out of the same material, the following complex coupling conditions may be successfully applied

$$\begin{array}{c} \tilde{\vec{Q}}_\rho = \tilde{\vec{Q}}_\rho^{\,C}, \quad \tilde{M}_{\rho\rho} = \tilde{M}_{\rho\rho}^{C} \\ \left(\tilde{\vec{Q}}_\rho = \vec{Q}_\rho + iEhc\vec{\varkappa}_\psi,\right. \\ \left.\tilde{M}_{\rho\rho} = M_{\rho\rho} + iEhc\varepsilon_{\psi\psi}\right). \end{array} \tag{12}$$

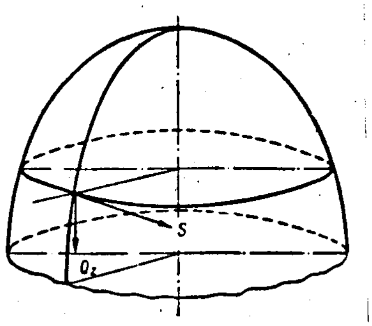

Figure 4

It is well known that the stresses Q_z and S (Figure 4) are fundamental stresses for symmetrically loaded shells of revolution, i.e., they do not depend on a simple edge effect. The basic results presented in a paper at the IV All Union Conference on the Theory of Shells and Plates (Erevan, October, 1962) indicated that [see (Ref. 6), page 244] similar static boundary values, which do not depend on the edge effect, may be obtained for an arbitrary non-asymptotic profile on a shell having a general form. This also pertains to deformation values. The compliance coefficients of the edge were also introduced (also for an arbitrary nonasymptotic profile on a shell having a general form).

Unfortunately, this entire study was only performed for customary exponential asymptotic behavior. The natural generalization of the results to the case of Bessel asymptotic behavior could facilitate the investigation of the concentration problem. In order to solve the "basic" problem, it is also advantageous to employ the theory of the complex variable functions (for shells produced by revolution of curves of the second order around their axes of symmetry). Particular attention must be called to the case when the fundamental stress state essentially differs from the zero-moment stress state. /317

In our opinion, the following problems must first be solved:

1. Stress concentrations in the vicinity of an elliptical hole in a sphere.

2. Stress concentrations in the vicinity of the same hole, but on a circular cylindrical shell. It is thus advantageous to process the results obtained based on the following four parameters: K (Gaussian curvature of the surface), $\frac{a+b}{2R_0}$ (relative dimensions of the hole), $\frac{h}{R_0}$ (thin-walled nature of the shell), and $|\Gamma_\psi| \frac{1}{H^2} \cdot \frac{\partial H}{\partial \rho}\Big|_{\rho = \rho_0}$ [geodesic curvature of the profile, see [(Ref. 6), page 71].

3. Stress concentration between two circular holes on spherical and cylindrical shells.

4. Stress concentration under the field conditions of holes with square and triangular partitions.

The solution of these problems would theoretically make it possible to transfer the results obtained to the case of shells and holes having a general form, and to provide recommendations of a structural nature.

These considerations, which naturally require practical verification, provide the basis for a study on stress concentration which was recently published in the Laboratory of Shell Theory of Leningrad State University.

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DYNAMIC STRESSES IN A THIN THERMOELASTIC PLATE WEAKENED BY A CIRCULAR HOLE /318

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The influence of a circular hole on stress distribution in an elastic plate under dynamic loading was investigated in (Ref. 5). The article (Ref. 2) studied the dynamic stress concentration around a circular hole when plane contraction waves pass through an elastic plate.

This study determines the steady stress state and temperature field around a circular hole in a thermoelastic plate, produced by the influence of concentrated force which is periodic in time (concentrated force, expansion center, etc.) at a certain point on the surface of the plate. The deformations and the stresses corresponding to them produced by these forces will be propagated in the plate in an undulating manner with a finite velocity. Allowance for scattering of mechanical energy in thermoelasticity equations leads to damping of these waves as they recede from the center.

The problem is solved in the linear formulation and under the assumption that the wavelength is large as compared with the plate thickness.

Formulation and general solution of the problem. Let us investigate an infinite thermoelastic plate with a circular hole having the radius R. A concentrated force is in operation at a certain point on the middle plane of this plate. The thermoelastic state of this plate may be described by the following equations

$$\begin{gathered}\nabla^2\bar{u} + (c^2-1)\,\mathrm{grad}\,\mathrm{div}\,\bar{u} - 2\alpha_t(c^2-1)\,\mathrm{grad}\,t = \frac{1}{c_2^2}\frac{\partial^2\bar{u}}{\partial\tau^2} - \frac{\bar{F}}{G};\\ \nabla^2 t - \frac{1}{\varkappa}\cdot\frac{\partial t}{\partial\tau} - \mu t - \frac{\gamma_0}{\varkappa}\cdot\frac{\partial}{\partial\tau}\,\mathrm{div}\,\bar{u} = -\mu t_c,\end{gathered} \tag{1}$$

where $\bar{\nabla}$ is a two-dimensional Hamiltonian operator; $\bar{u} = u\bar{k}_1 + v\bar{k}_2$ -- displacement vector; $\bar{F}$ -- mass force vector; t, t_c -- temperature of the plate and the surrounding medium; τ -- time; $c = c_1/c_2$; $c_1 = \sqrt{\frac{E}{\rho(1-\nu^2)}}$; $c_2 = \sqrt{\frac{G}{\rho}}$; /319

$\varkappa = \lambda/c_v(1+\varepsilon)$; $\mu = \alpha_p/h\lambda$; $\gamma_0 = \frac{\alpha_t E T_0}{(1-\nu)(1+\varepsilon)c_v}$; $\varepsilon = \frac{\alpha_t^2 E T_0}{c_v(1-2\nu)}\cdot\frac{1+\nu}{1-\nu}$, where h -- plate halfwidth; λ, α_p -- thermoconductivity and heat transfer coefficients; c_v -- specific heat per unit volume for a constant volume; α_t -- coefficient of linear expansion.

The stress tensor $\hat{\sigma}$ in two dimensions is related to the displacement vector $\bar{u}$ and the temperature by the following relationship:

$$\hat{\sigma} = \frac{E}{1-\nu^2}[\nu(\bar{\nabla}\bar{u}) - \alpha_t(1+\nu)t]\hat{I} + G(\bar{\nabla}\bar{u} + \bar{u}\bar{\nabla}), \tag{2}$$

where $\hat{I}$ is the unit tensor in two dimensions.

We shall try to determine the general solution of equations (1) in the following form (Ref. 3)

$$\bar{u} = \left(\nabla^2 - \frac{1}{c_1^2} \cdot \frac{\partial^2}{\partial \tau^2}\right)(\bar{\Phi}_1 + \bar{\Phi}_2) - \frac{c^2 - 1}{c^2} \operatorname{grad} \operatorname{div} (\bar{\Phi}_1 + \bar{\Phi}_2) + \\ + 2\varkappa_t \frac{c^2 - 1}{c^2} \operatorname{grad} \Psi, \quad t = \left(\nabla^2 - \frac{1}{c_1^2} \cdot \frac{\partial^2}{\partial \tau^2}\right) \Psi. \tag{3}$$

The functions $\bar{\Phi}_2$ and Ψ satisfy the following equations

$$\left[\left(\nabla^2 - \frac{1}{c_1^2} \cdot \frac{\partial^2}{\partial \tau^2}\right)\left(\nabla^2 - \frac{1}{\varkappa} \cdot \frac{\partial}{\partial \tau} - \mu\right) - \frac{\gamma}{\varkappa} \nabla^2 \frac{\partial}{\partial \tau}\right] \Psi = \\ = \frac{\gamma_0}{\varkappa c^2} \cdot \frac{\partial}{\partial \tau} \operatorname{div} \left(\nabla^2 - \frac{1}{c_2^2} \cdot \frac{\partial^2}{\partial \tau^2}\right) \bar{\Phi}_1 - \mu t_c; \quad \left(\nabla^2 - \frac{1}{c_2^2} \frac{\partial^2}{\partial \tau^2}\right) \bar{\Phi}_2 = 0 \\ [\gamma = \alpha_t (1 + \nu) \gamma_0], \tag{4}$$

and the function $\bar{\Phi}_1$ is a particular solution of the following equation

$$\left(\nabla^2 - \frac{1}{c_2^2} \cdot \frac{\partial^2}{\partial \tau^2}\right)\left(\nabla^2 - \frac{1}{c_1^2} \cdot \frac{\partial^2}{\partial \tau^2}\right) \bar{\Phi} = -\frac{F}{G}. \tag{5}$$

The forces applied at a point will be regarded as the limiting case of the body forces which are in operation in an unrestrictedly small vicinity of the point. By employing the device of the δ-function, we may reduce the problem of determining the stress state in a plate, produced by concentrated forces, to solving the equations (4) and (5) in the case of body forces which are given in the appropriate manner (Ref. 4).

The body force $\bar{F} = \bar{P}e^{i\omega\tau}\delta\,(x - a, y)$ corresponds to the concentrated force $\bar{P}e^{i\omega\tau}$ applied to the plate at the point (a, 0). The body force corresponding /320
to other concentrated forces may be written (Ref. 3) by the following expression

$$\bar{F} = -e^{i\omega\tau}\left[\left(p_{11}\frac{\partial}{\partial x} + p_{12}\frac{\partial}{\partial y} - m\frac{\partial}{\partial y}\right)\bar{k}_1 + \\ + \left(p_{12}\frac{\partial}{\partial x} + p_{22}\frac{\partial}{\partial y} + m\frac{\partial}{\partial x}\right)\bar{k}_2\right]\delta\,(x - a, y). \tag{6}$$

In the case of $p_{ij} = 0$, the mass force $\bar{F}$ corresponds to the concentrated moment at the point (a, 0). Assuming that $p_{12} = p_{21} = m = 0$, we obtain the body force equivalent to two forces which are in operation along the Ox and Oy axes. In the case of two equal forces $p_{11} = p_{22} = p$, we obtain the expansion center. If $p_{11} = p_{22} = m = 0$, and $p_{12} = p_{21} = q$, then the mass force $\bar{F}$ corresponds to a point singularity which may be called the displacement center.

Since thermoelastic dissipation leads to damping of the free oscillations, we shall investigate the steady state. Substituting the mass forces determined above in (5) in the case $t_c = 0$, we find that:

for a concentrated force

$$\bar{\Phi}_1 = \frac{i\bar{P}e^{i\omega\tau}}{4G\omega_1^2(1 - c^2)}\left[H_0^{(2)}(\omega_1 r_1) - H_0^{(2)}(\omega_2 r_1)\right]; \tag{7}$$

$$\Psi_1 = -\frac{\gamma_0}{\gamma}\cdot\frac{ie^{i\omega\tau}}{4G\omega_2^2}(\bar{P}\cdot\bar{\nabla})[H_0^{(2)}(\omega_1 r_1) + AH_0^{(1)}(\alpha r_1) - BH_0^{(1)}(\beta r_1)]; \tag{7}$$

for force factors (6)

$$\begin{gathered}\bar{\Phi}_1 = -\frac{ie^{i\omega\tau}}{4G\omega_1^2(1-c^2)}\left\{\left[\left(p_{11}\frac{\partial}{\partial x} + p_{12}\frac{\partial}{\partial y} - m\frac{\partial}{\partial y}\right)\bar{k}_1 + \right.\right.\\ \left.\left. + \left(p_{21}\frac{\partial}{\partial x} + p_{22}\frac{\partial}{\partial y} + m\frac{\partial}{\partial x}\right)\bar{k}_2\right]\left[H_0^{(2)}(\omega_1 r_1) - H_0^{(2)}(\omega_2 r_1)\right]\right\};\\ \Psi_1 = \frac{\gamma_0}{\gamma}\cdot\frac{ie^{i\omega\tau}}{4G\omega_2^2}\left(p_{11}\frac{\partial^2}{\partial x^2} + 2p_{12}\frac{\partial^2}{\partial x\partial y} + p_{22}\frac{\partial^2}{\partial y^2}\right)[H_0^{(2)}(\omega_1 r_1) + \\ + AH_0^{(1)}(\alpha r_1) - BH_0^{(1)}(\beta r_1)],\end{gathered} \tag{8}$$

where $H_0^{(1)}$, $H_0^{(2)}$ are the Hankel functions of the first and second type of zero order; $r_1 = \sqrt{r^2 + a^2 - 2ar\cos\theta}$ (the polar coordinates r, θ are introduced here);

$$\alpha^2 = \frac{1}{2}[\omega_1^2 - \mu - i(1+\gamma)\omega_0 + D];\quad \beta^2 = \frac{1}{2}[\omega_1^2 - \mu - i(1+\gamma)\omega_0 - D];$$

$$D = \sqrt{[\omega_1^2 - \mu - i(1+\gamma)\omega_0]^2 + 4\omega_1^2(\mu + i\omega_0)};$$

$$\omega_0 = \frac{\omega}{\varkappa};\quad \omega_1 = \frac{\omega}{c_1};\quad \omega_2 = \frac{\omega}{c_2};\quad A = \frac{1}{D}(\omega_1^2 - \beta^2);\quad B = \frac{1}{D}(\omega_1^2 - \alpha^2).$$

Expressions (7) and (8) describe diverging waves of three types: expansion waves, distortion waves, and thermal waves. When each of these waves collides with the boundary of a circular cavity, three types of waves are reflected. The reflected waves will be described by the following relationships /321

$$\begin{gathered}\bar{\Phi}_2 = \sum_{n=0}^{\infty}\left(k_1\frac{\partial}{\partial y} - k_2\frac{\partial}{\partial x}\right)(C_n\cos n\theta + C_n'\sin\theta)H_n^{(2)}(\omega_2 r)\cdot e^{i\omega\tau};\\ \Psi_2 = \sum_{n=0}^{\infty}[(A_n\cos n\theta + A_n'\sin\theta)H_n^{(1)}(\alpha r) + \\ + (B_n\cos n\theta + B_n'\sin n\theta)H_n^{(1)}(\beta r)\cdot e^{i\omega\tau}.\end{gathered} \tag{9}$$

The functions Φ_2 and Ψ_2 represent the solutions of the homogeneous equations (4). The unknown coefficients A_n, B_n, C_n, A_n', B_n', C_n' in (9) may be determined from the conditions at the boundary of the circular cavity ($r = R$). For a surface which is free of stress and through which heat is exchanged according to the Newton law, we have

$$\left.\begin{array}{ll}\sigma_{rr} = \sigma_{rr}^{(1)} + \sigma_{rr}^{(2)} = 0, & \sigma_{r\theta} = \sigma_{r\theta}^{(1)} + \sigma_{r\theta}^{(2)} = 0\\ \dfrac{\partial t}{\partial r} - kt = 0 & \left(k = \dfrac{\alpha_n}{\lambda}\right)\end{array}\right\}\text{ for } r = R. \tag{10}$$

Expansion center. Let the expansion center, which changes periodically in time, be in operation at the point (a, 0). Assuming that $p_{12} = p_{21} = m = 0$, $p_{11} = p_{22} = P$ in (8) and then substituting in (3), we obtain

$$\begin{gathered}\bar{u}^{(1)} = \frac{iPe^{i\omega\tau}}{4Gc^2D}\operatorname{grad}[\beta^2 BH_0^{(1)}(\beta r_1) - \alpha^2 AH_0^{(1)}(\alpha r_1)];\\ t^{(1)} = \frac{\gamma_0\omega_0 P}{4Gc^2D}[\alpha^2 H_0^{(1)}(\alpha r_1) - \beta^2 H_0^{(1)}(\beta r_1)]e^{i\omega\tau}.\end{gathered} \tag{11}$$

Assuming that $A_n' = B_n' = C_n' = 0$, we obtain the following from (9) and (3)

$$\bar{u}^{(2)} = \sum_{n=0}^{\infty}\left\{\omega_1^2(1-c^2)C_n\left(k_1\frac{\partial}{\partial y}-k_2\frac{\partial}{\partial x}\right)H_n^{(2)}(\omega_2 r)\sin n\theta + \right.$$
$$\left. + \alpha_t(1+\nu)\,\mathrm{grad}\,[A_nH_n^{(1)}(\alpha r)+B_nH_n^{(1)}(\beta r)]\cos n\theta\right\}e^{i\omega\tau}; \tag{12}$$
$$t^{(2)} = \sum_{n=0}^{\infty}[A_n(\omega_1^2-\alpha^2)H_n^{(1)}(\alpha r)+B_n(\omega^2-\beta^2)H_n^{(1)}(\beta r)]\cos n\theta\, e^{i\omega\tau}.$$

Utilizing the summation formulas of cylindrical functions (Ref. 1)

$$H_0^{(i)}\sqrt{r^2+a^2-2ar\cos\theta} =$$
$$= \begin{cases}\sum_0^{\infty}\varepsilon_nJ_n(r)H_n^{(i)}(a)\cos n\theta & r<a, \quad (i=1,2);\\ \sum_0^{\infty}\varepsilon_nH_n^{(i)}(r)J_n(a)\cos n\theta & r>a, \quad \varepsilon_n=\begin{cases}1, n=0;\\2, n>1,\end{cases}\end{cases} \tag{13}$$

by employing formulas (2) we obtain the following for $R \leq r \leq a$ /322

$$\sigma_{rr} = \frac{2Ge^{i\omega\tau}}{r^2}\sum_0^{\infty}\left\{\frac{\gamma}{\gamma_0}\left[A_nL_n(\alpha r)+B_nL_n(\beta r)\right] + \right.$$
$$\left. + \omega^2(1-c^2)C_nL_n(\omega_2 r)+p_n\right\}\cos n\theta;$$
$$\sigma_{\theta\theta}+\sigma_{rr} = -\frac{2Ge^{i\omega\tau}}{r^2}\sum_0^{\infty}\left\{s_n+\frac{\gamma}{\gamma_0}r^2[A_n(\omega^2-\alpha^2)H_n^{(1)}(\alpha r) + \right. \tag{14}$$
$$\left. + B_n(\omega^2-\beta^2)H_n^{(1)}(\beta r)\right\}\cos n\theta;$$
$$\sigma_{r\theta} = -\frac{2Ge^{i\omega\tau}}{r^2}\sum_0^{\infty}\left\{[A_nM_n(\alpha r)+B_nM_n(\beta r)]\frac{\gamma}{\gamma_0} + \right.$$
$$\left. + \omega_1^2(1-c^2)C_nM_n(\omega_2 r)+q_n\right\}\sin n\theta,$$

where

$$\left.\begin{aligned}
L_n(kr) &= \left[\left(n^2-n-\frac{1}{2}\omega_2^2r^2\right)H_n^{(1)}(kr)+krH_{n+1}^{(1)}(kr);\right.\\
M_n(kr) &= [n(n-1)H_n^{(1)}(kr)-nkrH_{n+1}^{(1)}(kr) \quad (k=\alpha,\beta);\\
L_n(\omega_2 r) &= [n(n-1)H_n^{(2)}(\omega_2 r)-n\omega_2 rH_{n+1}^{(2)}(\omega_2 r);\\
M_n'(\omega_2 r) &= \left[\left(n^2-n-\frac{1}{2}\omega_2^2r^2\right)H_n^{(2)}(\omega_2 r)+\omega_2 rH_{n+1}^{(2)}(\omega_2 r);\right.\\
p_n &= \frac{iP\varepsilon_n}{4G\omega_2^2}\left\{\beta^2B\left[\left(n^2-n-\frac{1}{2}\omega_2^2r^2\right)J_n(\beta r) + \right.\right.\\
&\left. + \beta rJ_{n+1}(\beta r)\right]H_n^{(1)}(\beta a)-\alpha^2A\left[\left(n^2-n-\frac{1}{2}\omega_2^2r^2\right)J_n(\alpha r) + \right.\\
&\left.\left. + \alpha rJ_{n+1}(\alpha r)\right]H_n^{(1)}(\alpha a)\right\};\\
s_n &= \frac{iP\varepsilon_nr^2}{4G\omega_2^2}\{\beta^2B(\omega_2^2-\beta^2)J_n(\beta r)H_n^{(1)}(\beta a) + \\
&+ \alpha^2A(\omega_2^2-\alpha^2)J_n(\alpha r)H_n^{(1)}(\alpha r)\};\\
q_n &= \frac{iP\varepsilon_n}{4G\omega_2^2}\{\alpha^2A[n(n-1)J_n(\alpha r)-n\alpha rJ_{n+1}(\alpha r)]H_n^{(1)}(\alpha a) - \\
&- \beta^2B[n(n-1)J_n(\beta r)-n\beta rJ_{n+1}(\beta r)]H_n^{(1)}(\beta a)\}.
\end{aligned}\right\} \tag{15}$$

The coefficients A_n, B_n, C_n may be determined from the equations

$$\frac{1}{\gamma_0}[A_n L_n(\alpha R) + B_n L_n(\beta R)] + \omega_1^2(1-c^2) L_n(\omega_2 R) C_n = -p_n;$$
$$\frac{1}{\gamma_0}[A_n M_n(\alpha R) + B_n M_n(\beta R)] + \omega_1^2(1-c^2) M_n(\omega_2 R) C_n = -q_n; \quad (16)$$
$$N_n(\alpha R) A_n + N_n(\beta R) B_n = t_n,$$

where /323

$$N_n(kR) = (\omega_1^2 - k)\left[\left(\frac{n}{R} - k\right) H_n^{(1)}(kR) - kH_{n+1}^{(1)}(kR)\right] \quad (k = \alpha, \beta);$$

$$t_n = \frac{\gamma_0 \omega_0 P \varepsilon_n}{4GDc^2}\left\{\alpha^2 H_n^{(1)}(\alpha a)\left[\left(\frac{n}{R} - k\right) J_n(\alpha R) - \alpha J_{n+1}(\alpha R)\right] - \right.$$
$$\left. - \beta^2 H_n^{(1)}(\beta a)\left[\left(\frac{n}{R} - k\right) J_n(\beta R) - \alpha J_{n+1}(\beta R)\right]\right\}.$$

For the region $r \geq R$, we obtain the stress expressions, exchanging the places of J_n and $H_n^{(1)}$ in (15) according to (13). Assuming that $\gamma_0 = 0$ in (15), we obtain the solution for an elastic plate.

Circular plate. Let us now investigate a circular plate which is loaded over the external profile ($r = R_2$) by the pressure $P_0 e^{i\omega_0 \tau}$ which is uniformly distributed. Due to the symmetry with respect to the center, there are no shear waves. Therefore, changing to dimensionless quantities in equation (1):

$$l = \frac{\varkappa}{c_1};\ \bar{u} = l\bar{u}_1;\ x = lx_1;\ y = ly_1;\ \tau =$$
$$= \frac{\varkappa}{c_1^2}\tau_1,\ t = \alpha_t(1+\nu)T;\ \mu_0 = \mu l^2;\ \sigma_{\theta\theta}^* = \frac{\sigma_{\theta\theta}(1-\nu^2)}{E};\ \omega_0 = \frac{c_1^2}{\varkappa}\omega,$$

we obtain the equation for determining the displacement potential $\Psi(\bar{u} = \text{grad}\ \Psi)$

$$\left[\left(\nabla^2 - \mu_0 - \frac{\partial}{\partial \tau_1}\right)\left(\nabla^2 - \frac{\partial^2}{\partial \tau_1^2}\right) - \gamma \frac{\partial}{\partial \tau_1}\nabla^2\right]\Psi = 0. \quad (17)$$

The solution of equation (17) will be

$$\Psi = [AJ_0(\alpha r) + BY_0(\alpha r) + A_1 J_0(\beta r) + B_1 Y_0(\beta r)] e^{i\omega\tau}, \quad (18)$$

where

$$\alpha^2, \beta^2 = \frac{1}{2}\{[\omega^2 - \mu_0 - i(1+\gamma)\omega] \pm$$
$$\pm \sqrt{[\omega^2 - \mu_0 - i(1+\gamma)\omega]^2 + 4\omega^2(\mu_0 + i\omega)}\}.$$

We may determine the unknown coefficients A, A_1, B, B_1 from the boundary conditions

$$\left.\begin{aligned} &\sigma_{rr}^* = 0;\ \frac{\partial T}{\partial r} - k_1 T = 0 \quad \text{for } r = R_1; \\ &\sigma_{rr}^* = -Pe^{i\omega\tau};\ \frac{\partial T}{\partial r} + k_2 T = 0 \quad \text{for } r = R_2. \end{aligned}\right\} \quad (19)$$

We have the following stresses in a thermoelastic plate

$$\sigma_{\theta\theta}^* = \frac{PR_2^2}{r^2}\frac{\Delta_1}{\Delta} e^{i\omega\tau}, \quad (20)$$

where /324

$$\Delta = \begin{vmatrix} F_2(\alpha) & E_2(\alpha) & F_2(\beta) & E_2(\beta) \\ F_1(\alpha) & E_1(\alpha) & F_1(\beta) & E_1(\beta) \\ N_2(\alpha) & M_2(\alpha) & N_2(\beta) & M_2(\beta) \\ N_1(\alpha) & M_1(\alpha) & N_1(\beta) & M_1(\beta) \end{vmatrix}; \quad \Delta_1 = \begin{vmatrix} S(\alpha r) & Q(\alpha r) & S(\beta r) & Q(\beta r) \\ F_1(\alpha) & E_1(\alpha) & F_1(\beta) & E_1(\beta) \\ N_2(\alpha) & M_2(\alpha) & N_2(\beta) & M_2(\beta) \\ N_1(\alpha) & M_1(\alpha) & N_1(\beta) & M_1(\beta) \end{vmatrix};$$

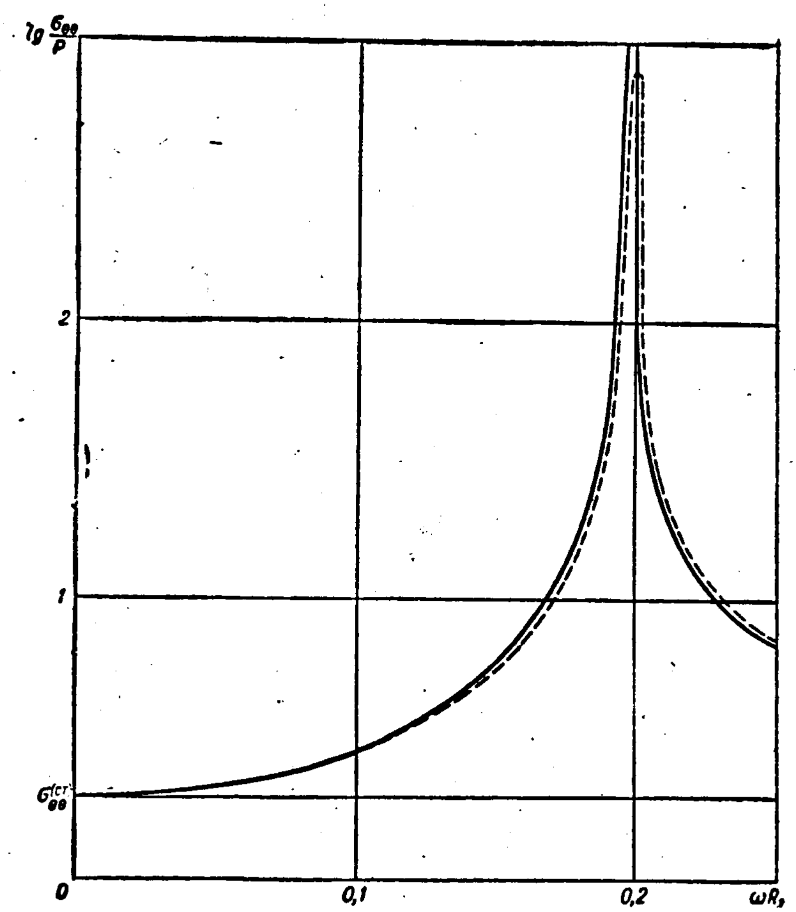
$$S(kr) = \left(k^2 - \frac{c^2\omega^2}{2}\right) rJ_0(kr) - kJ_1(kr);$$

$$Q(kr) = \left(k^2 - \frac{c^2\omega}{2}\right) rY_0(kr) - kY_1(kr);$$

$$F_i(k) = kR_iJ_1(kR_i) - \frac{c\omega}{2} R_i^2 J_0(kR_i);$$

$$E_i(k) = kR_iY_1(kR_i) - \frac{c\omega}{2} R_i^2 Y_0(kR_i);$$

$$N_i(k) = (\omega^2 - k^2)\,[kJ_1(kR_i) - (-1)^i k_i J_0(kR_i)] \quad (k = \alpha, \beta);$$
$$M_i(k) = (\omega^2 - k^2)\,[kY_1(kR_i) - (-1)^i k_i Y_0(kR_i)] \quad (i = 1, 2).$$



In the case of γ_0, we obtain the following stresses in an elastic plate /325

$$\sigma_{\theta\theta}^* = \frac{PR_2}{r\Delta}\left\{\left[Y_1(\omega R_1) - \frac{\omega R_1}{1-\nu} Y_0(\omega R_1)\right]\left[\frac{\nu}{1-\nu}\omega r J_0(\omega r) + J_1(\omega r)\right] - \right. \tag{21}$$
$$\left. - \left[J_1(\omega R_1) - \frac{\omega R_1}{1-\nu} J_0(\omega R_1)\right]\left[\frac{\nu\omega r}{1-\nu} Y_0(\omega r) + Y_1(\omega r)\right]\right\} e^{i\omega\tau},$$

where

$$\Delta = \left[Y_1(\omega R_1) - \frac{\omega R_1}{1-\nu} Y_0(\omega R_1)\right]\left[J_1(\omega R_2) - \frac{\omega R_2}{1-\nu} J_0(\omega R_2)\right] - \tag{22}$$

$$-\left[Y_1(\omega R_2)-\frac{\omega R_2}{1-\nu}Y_0(\omega R_2)\right]\left[J_1(\omega R_1)-\frac{\omega R_1}{1-\nu}J_0(\omega R_1)\right]. \tag{22}$$

Passing to the limit in (22) in the case $\omega \to 0$, we obtain a result which corresponds to the static problem.

The figure shows the stress distribution $\sigma^*_{\theta\theta}$ on the profile $r = R_1$ of an aluminum plate as a function of the frequency ω. The solid line in the figure designates the stress $\sigma_{\theta\theta}$ in an elastic plate, and the dashed line designates the stress in a thermoelastic plate.

The computations were performed for $\tau = 0$ with the following parameter values: $\nu = 0, 3$, $\frac{R2}{R1} = 10$, $\gamma = 0.01$. In the case $\omega \to 0$, the dynamic stresses decrease, asymptotically approaching the static value $\sigma^{(st)}_{\theta\theta}$. In the case $\omega \approx 0.198$, major resonance occurs in an elastic plate, whereas the amplitude remains limited in a thermoelastic plate, due to thermoelastic damping.

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INFLUENCE OF CONCENTRATED EFFECTS ON SHALLOW SHELLS /326

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Comparatively little research (Ref. 1, 2, 4, 10, 11) has been devoted to the problem of the influence of concentrated forces on shallow shells. The authors in these studies confine themselves to only examining shells of a particular type, subjected to the influence of a normal concentrated force. In these studies, the solution of the problem indicated above is presented in double or single trigonometric series, and due to this fact it is difficult to calculate certain internal stresses due to the poor convergence of the series.

The influence of concentrated bending moments and tangential forces has not been investigated in the literature. By employing the two-dimensional integral Fourier transformation and the theory of generalized functions, we obtained a particular solution of a system of differential equilibrium equations of shallow shells having constant curvature, subjected to the influence of any concentrated effects. In the case of a shallow spherical and cylindrical shell, the solution obtained has a closed form.

Equilibrium equations. The equilibrium equations of a shallow shell having constant curvature may be represented in the following form in rectangular coordinates x, y

$$\left.\begin{aligned}&\frac{\partial^2 u}{\partial x^2}+\frac{1-\nu}{2}\cdot\frac{\partial^2 u}{\partial y^2}+\frac{1+\nu}{2}\cdot\frac{\partial^2 v}{\partial x\partial y}-\frac{\lambda+\nu}{R_2}\cdot\frac{\partial w}{\partial x}=-\frac{1-\nu^2}{Eh}X;\\&\frac{1+\nu}{2}\cdot\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}+\frac{1-\nu}{2}\cdot\frac{\partial^2 v}{\partial x^2}-\frac{1+\lambda\nu}{R^2}\cdot\frac{\partial w}{\partial y}=-\frac{1-\nu^2}{Eh}Y;\\&\nabla^4 w+\frac{12}{h^2R_2^2}[1+\lambda\nu+\lambda(\lambda+\nu)]\,w-\frac{12}{h^2R_2}\Big[(\lambda+\nu)\frac{\partial u}{\partial x}+\\&\qquad+(1+\lambda\nu)\frac{\partial v}{\partial y}\Big]=\frac{Z}{D}.\end{aligned}\right\}\tag{1}$$

We may determine the components of the internal stresses according to the following formulas (Ref. 7) /327

$$\left.\begin{aligned}&T_1=\frac{Eh}{1-\nu^2}\left(\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}-\frac{\lambda+\nu}{R_2}w\right);\ M_1=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right);\\&T_2=\frac{Eh}{1-\nu^2}\left(\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}-\frac{1+\lambda\nu}{R^2}w\right);\ M_2=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right);\\&S=\frac{Eh}{2(1+\nu)}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right);\ H=-D(1-\nu)\frac{\partial^2 w}{\partial x\partial y};\\&Q_1=-D\frac{\partial}{\partial x}\nabla^2 w;\quad Q_2=-D\frac{\partial}{\partial y}\nabla^2 w.\end{aligned}\right\}\tag{2}$$

We shall employ the following notation below: u, v, w, X, Y, Z -- components of displacement and surface loading; $\lambda = R_2/R_1$; R_1 and R_2 -- main radii of curvature; $D = \frac{Eh^3}{12(1-\nu^2)}$; h -- shell thickness; ν -- Poisson coefficient. The general solution (1) consists of the solution of the homogeneous system obtained from (1) and a particular solution.

The problem of finding a solution of the homogeneous system will not be examined here. We would only like to indicate that (1) may be reduced to a single differential equation of the eigth order by introducing the displacement function.

Let us obtain the particular solution of system (1). Subjecting (1) and (2) to a two-dimensional integral Fourier transformation (Ref. 8)

$$\bar{f}(\xi, \eta) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} f(x, y)\exp i(\xi x + \eta y)\,dxdy,$$

we obtain

(a) For (1):

$$\left.\begin{aligned} &-\left(\xi^2 + \frac{1-\nu}{2}\eta^2\right)\bar{u} - \frac{1+\nu}{2}\xi\eta\bar{v} + \frac{\lambda+\nu}{R_2}i\xi\bar{w} = -\frac{1-\nu^2}{Eh}\bar{X};\\ &-\frac{1+\nu}{2}\xi\eta\bar{u} - \left(\eta^2 + \frac{1-\nu}{2}\xi^2\right)\bar{v} + \frac{1+\lambda\nu}{R_2}i\eta\bar{w} = -\frac{1-\nu^2}{Eh}\bar{Y};\\ &(\xi^2+\eta^2)^2\bar{w} + \frac{12}{R_2^2h^2}[1+\lambda\nu+\lambda(\lambda+\nu)]\bar{w} +\\ &+\frac{12}{h^2R_2}[(\lambda+\nu)i\xi\bar{u} + (1+\lambda\nu)i\eta\bar{v}] = \frac{\bar{Z}}{D}\end{aligned}\right\} \qquad (3)$$

(all the functions with the dash above indicate the Fourier transform of the corresponding function);

(b) For (2):

$$\left.\begin{aligned} &\bar{T}_1 = -\frac{Eh}{1-\nu^2}\left[i\xi\bar{u} + \nu i\eta\bar{v} + \frac{\lambda+\nu}{R_2}\bar{w}\right]; && \bar{M}_1 = D(\xi^2+\nu\eta^2)\bar{w};\\ &\bar{T}_2 = -\frac{Eh}{1-\nu^2}\left[i\eta\bar{v} + \nu i\xi\bar{u} + \frac{1+\lambda\nu}{R_2}\bar{w}\right]; && \bar{M}_2 = D(\eta^2+\nu\xi^2)\bar{w};\\ &\bar{S} = -\frac{Eh}{2(1+\nu)}[i\xi\bar{v} + i\eta\bar{u}]; && \bar{H} = D(1-\nu)\xi\eta\bar{w};\\ &\bar{Q}_1 = -Di\xi(\xi^2+\eta^2)\bar{w}; && \bar{Q}_2 = -Di\eta(\xi^2+\eta^2)\bar{w}.\end{aligned}\right\} \qquad (4)$$

Let us now examine a series of particular cases: /328

A. X = Y = 0.

Determining $\bar{u}$, $\bar{v}$, $\bar{w}$ from (3) and substituting their values in (4), after certain simplifications we have

$$\bar{u} = \frac{\bar{Z}}{R_2D}\cdot\frac{1}{\Delta}i\xi[(\lambda+\nu)\xi^2 + (2\lambda+\lambda\nu-1)\eta^2]; \qquad (5a)$$

$$\bar{v} = \frac{\bar{Z}}{R_2D}\cdot\frac{1}{\Delta}i\eta[(2+\nu-\lambda)\xi^2 + (1+\lambda\nu)\eta^2]; \qquad (5b)$$

$$\bar{w} = \frac{\bar{Z}}{D}\cdot\frac{1}{\Delta}(\xi^2+\eta^2)^2; \qquad (5c)$$

$$\bar{T}_1 = -\frac{Eh}{R_2D}\cdot\frac{\bar{Z}}{\Delta}\eta^2(\xi^2+\lambda\eta^2); \quad \bar{M}_1 = \frac{\bar{Z}}{\Delta}(\xi^2+\eta^2)^2(\xi^2+\nu\eta^2); \qquad (5d)$$

$$\bar{T}_2 = -\frac{Eh}{R_2D}\cdot\frac{\bar{Z}}{\Delta}\xi^2(\xi^2+\lambda\eta^2); \quad \bar{M}_2 = \frac{\bar{Z}}{\Delta}(\xi^2+\eta^2)^2(\nu\xi^2+\eta^2); \qquad (5e)$$

$$\bar{S} = \frac{Eh}{R_2D}\cdot\frac{\bar{Z}}{\Delta}\xi\eta(\xi^2+\lambda\eta^2); \quad \bar{H} = (1-\nu)\frac{\bar{Z}}{\Delta}\xi\eta(\xi^2+\eta^2)^2; \qquad (5f)$$

$$\bar{Q}_1 = -\frac{\bar{Z}}{\Delta}i\xi(\xi^2+\eta^2)^3; \quad \bar{Q}_2 = -\frac{\bar{Z}}{\Delta}i\eta(\xi^2+\eta^2)^3, \qquad (5g)$$

where

$$\Delta=(\xi^2+\eta^2)^4+k^4(\xi^2+\lambda\eta^2)^2;$$
$$k^4=\frac{12(1-\nu^2)}{h^2R_2^2};\quad i=\sqrt{-1}.$$

B. Y = Z = 0.

We then have

$$\bar{u}=\frac{\bar{X}}{R_2^2D}\cdot\frac{\xi^2[(\lambda+\nu)\xi^2+(2\lambda+\lambda\nu-1)\eta^2]^2}{\Delta\cdot(\xi^2+\eta^2)^2}+\frac{2(1+\nu)}{Eh}\bar{X}\frac{\eta^2+\frac{1-\nu}{2}\xi^2}{(\xi^2+\eta^2)^2};\qquad(6a)$$

$$\bar{v}=\frac{\bar{X}}{R_2^2D}\cdot\frac{\xi\eta[(2+\nu-\lambda)\xi^2+(1+\lambda\nu)\eta^2][(\lambda+\nu)\xi^2+(2\lambda+\lambda\nu-1)\eta^2]}{\Delta\cdot(\xi^2+\eta^2)^2}-\frac{(1+\nu)^2}{Eh}\bar{X}\frac{\xi\eta}{(\xi^2+\eta^2)^2};\qquad(6b)$$

$$\bar{w}=-\frac{\bar{X}}{R_2D}\cdot\frac{i\xi[(\lambda+\nu)\xi^2+(2\lambda+\lambda\nu-1)\eta^2]}{\Delta};\qquad(6c)$$

$$\bar{T}_1=\frac{Eh}{R_2^2D}\times\frac{\bar{X}}{\Delta}\cdot\frac{i\xi\eta^2\{(\lambda+\nu)\xi^4+[(2\lambda+\lambda\nu-1)+\lambda(\lambda+\nu)]\xi^2\eta^2+\lambda(2\lambda+\lambda\nu-1)\eta^4\}}{(\xi^2+\eta^2)^2}-\bar{X}\cdot\frac{i\xi[\xi^2+(2+\nu)\eta^2]}{(\xi^2+\eta^2)^2};\qquad(6d)$$

$$\bar{T}_2=\frac{Eh}{R_2^2D}\times\frac{\bar{X}}{\Delta}\frac{i\xi^3[(\lambda+\nu)\xi^4+\lambda(2\lambda+\lambda\nu-1)\eta^4+(\lambda^2+2\lambda\nu+2\lambda-1)\xi^2\eta^2]}{(\xi^2+\eta^2)^2}-\bar{X}\cdot\frac{i\xi(\nu\xi^2-\eta^2)}{(\xi^2+\eta^2)^2};\qquad(6e)$$ /329

$$\bar{S}=-\frac{Eh}{R_2^2D}\cdot\frac{\bar{X}}{\Delta}\times\frac{i\xi^2\eta[(\lambda+\nu)\xi^4+(\lambda^2+2\lambda\nu+2\lambda-1)\xi^2\eta^2+\lambda(2\lambda+\lambda\nu-1)\eta^4]}{(\xi^2+\eta^2)^2}-\bar{X}\cdot\frac{i\eta(\eta^2-\nu\xi^2)}{(\xi^2+\eta^2)^2}.\qquad(6f)$$

The expressions for $\bar{M}_1$, $\bar{M}_2$, $\bar{H}$, $\bar{Q}_1$, $\bar{Q}_2$ may be readily determined from (4) (6a) - (6f).

C. X = Z = 0.

We thus have

$$\bar{u}=\frac{\bar{Y}}{R_2^2D}\cdot\frac{\xi\eta[(2\lambda+\lambda\nu-1)\eta^2+(\lambda+\nu)\xi^2][(2+\nu-\lambda)\xi^2+(1+\lambda\nu)\eta^2]}{\Delta(\xi^2+\eta^2)^2}-\frac{(1+\nu)^2}{Eh}\bar{Y}\frac{\xi\eta}{(\xi^2+\eta^2)^2};\qquad(7a)$$

$$\bar{v}=\frac{\bar{Y}}{R_2^2D}\cdot\frac{\eta^2[(2+\nu-\lambda)\xi^2+(1+\lambda\nu)\eta^2]^2}{\Delta(\xi^2+\eta^2)^2}+$$

$$+\frac{2(1+\nu)}{Eh}\bar{Y}\frac{\xi^2+\frac{1-\nu}{2}\eta^2}{(\xi^2+\eta^2)^2}; \tag{7b}$$

$$\bar{w}=-\frac{\bar{Y}}{R_2D}\cdot\frac{i\eta[(2+\nu-\lambda)\xi^2+(1+\lambda\nu)\eta^2]}{\Delta}; \tag{7c}$$

$$\bar{T}_1=\frac{Eh}{R_2^2D}\cdot\frac{\bar{Y}}{\Delta}\frac{i\eta^3[(2+\nu-\lambda)\xi^4+(1+2\lambda+2\lambda\nu-\lambda^2)\xi^2\eta^2+\lambda(1+\lambda\nu)\eta^4]}{(\xi^2+\eta^2)^2}-$$
$$-\bar{Y}\frac{i\eta(\nu\eta^2-\xi^2)}{(\xi^2+\eta^2)^2}; \tag{7d}$$

$$\bar{T}_2=\frac{Eh}{R_2^2D}\cdot\frac{\bar{Y}}{\Delta}\frac{i\eta\xi^2\{(2+\nu-\lambda)\xi^4+[\lambda(2+\nu-\lambda)+(1+\lambda\nu)]\xi^2\eta^2+\lambda(1+\lambda\nu)\eta^4\}}{(\xi^2+\eta^2)^2}-$$
$$-\bar{Y}\frac{i\eta[\eta^2+(2+\nu)\xi^2]}{(\xi^2+\eta^2)^2}; \tag{7e}$$

$$\bar{S}=-\frac{Eh}{R_2^2D}\cdot\frac{\bar{Y}}{\Delta}\cdot\frac{i\xi\eta^2[(2+\nu-\lambda)\xi^4+(1+2\lambda-\lambda^2+2\lambda\nu)\xi^2\eta^2+\lambda(1+\lambda\nu)\eta^4]}{(\xi^2+\eta^2)^2}-$$
$$-\bar{Y}\frac{i\xi(\xi^2-\nu\eta^2)}{(\xi^2+\eta^2)^2}. \tag{7f}$$

We may readily determine the values of $\bar{M}_1$, $\bar{M}_2$, $\bar{H}$, $\bar{Q}_1$, $\bar{Q}_2$ from (4) and (7a) - (7f). Let us represent (Ref. 9, 12) the unit concentrated force by means of the generalized function (Ref. 3) /330

$$P=\delta(x-x_0)\delta(y-y_0), \tag{8}$$

where δ is the delta function; x_0, y_0 -- coordinates of the point at which the concentrated force is applied. For purposes of simplicity, we shall assume $x_0 = y_0 = 0$.

Employing the inverse transform formula for the Fourier transformation (Ref. 8)

$$f(x, y)=\frac{1}{2\pi}\int\int_{-\infty}^{\infty}\bar{f}(\xi, \eta)\exp[-i(\xi x+\eta y)]\,d\xi d\eta, \tag{9}$$

we obtain the expressions for the inverse transforms u, v, w, ..., Q_2. However, this is difficult to accomplish in the general case. Therefore, let us investigate several particular problems arising from the method under consideration.

Shallow spherical shell. In this case, we must set $\lambda = 1$ in (1) - (7f). Applying the inversion formula (9) to the expressions obtained and the theorem of residues, after an entire series of simplifications we obtain:

(a) From (5a) - (5f)

$$u=-\frac{1+\nu}{2\pi Eh}R\left[\frac{x}{r^2}+\frac{kx}{r}u_0'(kr)\right]; \tag{10a}$$

$$v=-\frac{1+\nu}{2\pi Eh}R\left[\frac{y}{r^2}+\frac{ky}{r}u_0'(kr)\right]; \tag{10b}$$

$$w=-\frac{1}{2\pi k^2D}v_0(kr); \tag{10c}$$

$$T_1 = -\frac{R}{2\pi}\left\{\frac{x^2-y^2}{r^4} + \frac{k}{\sqrt{2}}\left[\frac{x^2}{r^3}(u_1+v_1) + \frac{y^2}{r^2}k(u_1'+v_1')\right]\right\}; \tag{10d}$$

$$T_2 = -\frac{R}{2\pi}\left\{-\frac{x^2-y^2}{r^4} + \frac{k}{\sqrt{2}}\left[\frac{y^2}{r^3}(u_1+v_1) + \frac{x^2}{r^2}k(u_1'+v_1')\right]\right\}; \tag{10e}$$

$$S = -\frac{R}{\pi}\left[\frac{xy}{r^4} - \frac{k^2}{2}\cdot\frac{xy}{r^2}v_2\right]; \tag{10f}$$

$$M_1 = \frac{1-\nu}{2\pi}\cdot\frac{1}{k\sqrt{2}}\left[\frac{x^2}{r^3}(u_1-v_1) + \frac{y^2}{r^2}k(u_1'-v_1')\right] + \frac{1}{2\pi}u_0; \tag{10g}$$

$$M_2 = -\frac{1-\nu}{2\pi}\cdot\frac{1}{k\sqrt{2}}\left[\frac{x^2}{r^3}(u_1-v_1) + \frac{y^2}{r^2}k(u_1'-v_1')\right] + \frac{\nu}{2\pi}u_0; \tag{10h}$$

$$H = -\frac{1-\nu}{2\pi}\cdot\frac{xy}{r^2}u_2; \quad Q_1 = \frac{1}{2\pi}\cdot\frac{k}{\sqrt{2}}\cdot\frac{x}{r}(u_1+v_1);$$

$$Q_2 = \frac{1}{2\pi}\cdot\frac{k}{\sqrt{2}}\cdot\frac{y}{r}(u_1+v_1); \tag{10i}$$

(b) From (6a) - (6f) /331

$$u = -\frac{(1+\nu)^2}{2\pi Eh}\left[\frac{y^2}{kr^3}v_0' - \frac{x^2}{r^2}\cdot\frac{1}{\sqrt{2}}(u_1'-v_1')\right] - \frac{1+\nu}{\pi Eh}\ln r; \tag{11a}$$

$$v = \frac{(1+\nu)^2}{2\pi Eh}\left[\frac{xy}{kr^3}v_0' + \frac{xy}{r^2}\cdot\frac{1}{\sqrt{2}}(u_1-v_1')\right]; \tag{11b}$$

$$w = \frac{1+\nu}{2\pi Eh}R\left[\frac{x}{r^2} + \frac{kx}{r}u_0'\right]; \tag{11c}$$

$$T_1 = -\frac{1+\nu}{2\pi}\left[\frac{x^3-xy^2}{r^4}u_2 + \frac{xy^2}{r^3}ku_2'\right] - \frac{1}{2\pi}\cdot\frac{x}{r^2}; \tag{11d}$$

$$T_2 = -\frac{1+\nu}{8\pi}\frac{k}{\sqrt{2}}\left[\frac{x^3-3xy^2}{r^3}(u_3+v_3) - \frac{3x}{r}(u_1+v_1)\right] + \frac{1}{2\pi}\cdot\frac{x}{r^2}; \tag{11e}$$

$$S = \frac{1+\nu}{2\pi}\left[\frac{y^3-x^2y}{r^4}u_2 + \frac{x^2y}{r^3}ku_2'\right] - \frac{1}{2\pi}\cdot\frac{y}{r^2}; \tag{11f}$$

$$M_1 = -\frac{1-\nu^2}{2\pi k^2R}\left[\frac{x^3-xy^2}{r^4}v_2 + \frac{xy^2}{r^3}kv_2'\right] + \frac{1+\nu}{2\pi kR}\frac{x}{r\sqrt{2}}(u_1-v_1) + {}$$
$$+ \frac{1-\nu^2}{\pi k^4R}\cdot\frac{x^3-3xy^2}{r^6}; \tag{11g}$$

$$M_2 = \frac{1-\nu^2}{2\pi^2R}\left[\frac{x^3-xy^2}{r^4}v^2 + \frac{xy^2}{r^3}kv_2'\right] + {}$$
$$+ \frac{\nu(1+\nu)}{2\pi kR}\cdot\frac{x}{r\sqrt{2}}(u_1-v_1) - \frac{1-\nu^2}{\pi k^4R}\cdot\frac{x^3-3xy^2}{r^6}; \tag{11h}$$

$$H = \frac{1-\nu^2}{2\pi k^2R}\left[\frac{y^3-x^2y}{r^4}v_2 + \frac{x^2y}{r^3}kv_2'\right] + \frac{1-\nu^2}{\pi k^4R}\cdot\frac{3x^2y-y^3}{r^6}; \tag{11i}$$

$$Q_1 = \frac{1+\nu}{2\pi kR}\cdot\frac{1}{\sqrt{2}}\left[\frac{y^2}{r^3}(u_1-v_1) + \frac{x^2}{r^2}k(u_1'-v_1')\right]; \tag{11j}$$

$$Q_2 = -\frac{1+\nu}{2\pi R}\frac{xy}{r^2}u_2; \tag{11k}$$

(c) From (7a) - (7f)

$$u = \frac{(1+\nu)^2}{2\pi Eh}\left[\frac{xy}{kr^3}v_0' + \frac{xy}{r^2}\frac{1}{\sqrt{2}}(u_1'-v_1')\right]; \tag{12a}$$

$$v = -\frac{(1+\nu)^2}{2\pi Eh}\left[\frac{y^2}{kr^3}v_0' - \frac{x^2}{r^2}\cdot\frac{1}{\sqrt{2}}(u_1'-v_1')\right] - \frac{1+\nu}{\pi Eh}\ln r; \tag{12b}$$

$$w = \frac{1+\nu}{2\pi Eh}R\left(\frac{y}{r^2} + \frac{ky}{r}u_0'\right); \tag{12c}$$

$$T_1 = \frac{1+\nu}{8\pi} \cdot \frac{k}{\sqrt{2}}\left[\frac{3x^2y - y^3}{r^3}(u_3 + v_3) + \frac{3y}{r}(u_1 + v_1)\right] + \frac{1}{2\pi} \cdot \frac{y}{r^2}; \tag{12d}$$

$$T_2 = -\frac{1+\nu}{2\pi}\left(\frac{y^3 - x^2y}{r^4} u_2 + \frac{x^2y}{r^3} ku_2'\right) - \frac{1}{2\pi} \cdot \frac{y}{r^2}; \tag{12e}$$

$$S = \frac{1+\nu}{2\pi}\left(\frac{x^3 - xy^2}{r^4} u_2 + \frac{xy^2}{r^3} ku_2'\right) - \frac{1}{2\pi} \cdot \frac{x}{r^2}; \tag{12f}$$

$$M_1 = \frac{1-\nu^2}{\pi k^4 R} \cdot \frac{3x^2y - y^3}{r^6} - \frac{1-\nu^2}{8\pi R} \cdot \frac{1}{k\sqrt{2}}\left[\frac{3y}{r}(u_1 - v_1) + \right.$$
$$\left. + \frac{3x^2y - y^3}{r^3}(u_3 - v_3)\right] + \frac{1+\nu}{2\pi R} \frac{1}{k\sqrt{2}} \frac{y}{r}(u_1 - v_1); \tag{12g}$$ /332

$$M_2 = \frac{1-\nu^2}{\pi k^4 R} \frac{y^3 - 3x^2y}{r^6} + \frac{1-\nu^2}{8\pi R} \frac{1}{k\sqrt{2}}\left[\frac{3y}{r}(u_1 - v_1) + \right.$$
$$\left. + \frac{3x^2y - y^3}{r^3}(u_3 - v_3)\right] + \frac{\nu(1+\nu)}{2\pi R} \frac{1}{k\sqrt{2}} \frac{y}{r}(u_1 - v_1); \tag{12h}$$

$$H = -\frac{1-\nu^2}{\pi k^4 R} \cdot \frac{x^3 - 3xy^2}{r^6} + \frac{1-\nu^2}{2\pi k^2 R}\left[\frac{x^3 - xy^2}{r^3} v_2 + \frac{xy^2}{r^3} kv_2'\right]; \tag{12i}$$

$$Q_1 = -\frac{1+\nu}{2\pi R} \cdot \frac{xy}{r^2} u_2, \tag{12j}$$

$$Q_2 = -\frac{1+\nu}{2\pi k R} \cdot \frac{1}{\sqrt{2}}\left[\frac{x^2}{r^3}(u_1 - v_1) + \frac{y^2}{r^2} k(u_1' - v_1')\right]. \tag{12k}$$

The following notation is introduced in (10a) - (12i)

$$u_n(z) + iv_n(z) = i^{-n} K_n(z\sqrt{i}); \quad z = kr; \quad r^2 = x^2 + y^2,$$

where $K_n(z\sqrt{i})$ is a modified Bessel function of the second kind (Ref. 5). The theory of generalized functions (Ref. 3) is employed here to calculate certain divergent integrals. We shall show that

$$\int_0^\infty e^{-\eta x} \frac{\cos \eta y}{\eta} d\eta = -\ln r + C, \tag{13}$$

where C is an Euler constant.

Let us investigate the generalized function (Ref. 3)

$$\ln x_+ = \begin{cases} \ln x & x > 0; \\ 0 & x < 0. \end{cases} \tag{14}$$

The generalized function x^{-1} will be the derivative of this function. Further simple transformations (Ref. 3) lead us to (13).

Employing the well known results of the theory of generalized functions and fundamental solutions of an elliptical system of differential equations with constant coefficients, we may readily obtain a solution of the problem in the case when a unit bending moment acts on the shell.

It is known (Ref. 9, 12) that the unit bending moment may be regarded as a derivative of the delta function. Consequently, in this case a solution is obtained by simple differentiation of (10a) - (12i) with respect to the corresponding variable. Regarding (10a) - (12i) as influence functions, we obtain the solution for case of arbitrary loading.

In the case when symmetrical loading acts on the shell, the method of integral Hankel transformation (Ref. 8) represents a more effective method for obtaining the particular solution. In (13) this method is employed to obtain particular solutions for the case of loading of a shallow spherical shell by loads applied over the circular regions. /333

Shallow cylindrical shell. We must set $\lambda = 0$ in (1) - (7f). Procedures similar to the preceding ones lead to the following results:

(a) From (5a) - (5g)

$$u = -\frac{1+\nu}{8\pi k^3 RD}\left[f_4(u_0+v_0) - f_3(u_0-v_0)\right] - \frac{1}{4\pi k^2 RD}\int_0^x (f_1 v_0 + + f_2 u_0)\,dx; \tag{15a}$$

$$v = \frac{1+\nu}{8\pi k^3 RD}\int_0^x \frac{y}{r}(f_3 u_1 + f_4 v_1)\,dx - \frac{1}{4\pi k^2 RD}\int_0^y (f_1 v_0 - f_2 u_0)\,dy - - \frac{1}{4\pi k RD}\int_0^x\int_0^y \left[f_3(u_0+v_0) - f_4(v_0-u_0)\right]dx\,dy; \tag{15b}$$

$$w = \frac{1}{4\pi k D}\int_0^x \left[f_3(u_0+v_0) - f_4(v_0-u_0)\right]dx; \tag{15c}$$

$$T_1 = \frac{Eh}{8\pi k^2 RD}\left[(f_1 v_0 + f_2 u_0) + \frac{x}{r}(f_3 u_1 - f_4 v_1)\right]; \tag{15d}$$

$$T_2 = \frac{Eh}{8\pi k^2 RD}\left[(f_1 v_0 + f_2 u_0) - \frac{x}{r}(f_3 u_1 - f_4 v_1)\right]; \tag{15e}$$

$$S = \frac{Eh}{8\pi k^2 RD}\cdot\frac{y}{r}(f_3 u_1 - f_4 v_1); \tag{15f}$$

$$M_1 = \frac{1+\nu}{4\pi}(f_1 u_0 - f_2 v_0) - \frac{1-\nu}{4\pi}\cdot\frac{x}{r}(f_3 v_1 + f_4 u_1); \tag{15g}$$

$$M_2 = \frac{1+\nu}{4\pi}(f_1 u_0 - f_2 v_0) + \frac{1-\nu}{4\pi}\cdot\frac{x}{r}(f_3 v_1 + f_4 u_1); \tag{15h}$$

$$H = \frac{1-\nu}{4\pi}\cdot\frac{y}{r}(f_3 v_1 + f_4 u_1); \tag{15i}$$

$$Q_1 = \frac{k}{4\pi}\left\{\left[f_3(u_0-v_0) + f_4(u_0+v_0)\right] + \frac{x}{r}\left[f_1(u_1+v_1) + + f_2(u_1-v_1)\right]\right\}; \tag{15j}$$

$$Q_2 = \frac{k}{4\pi}\cdot\frac{y}{r}\left[f(u_1+v_1) + f_2(u_1-v_1)\right]; \tag{15k}$$ /334

(b) From (6a) - (6f)

$$u = \frac{(3-\nu)(1+\nu)}{4\pi Eh}(f_1 u_0 - f_2 v_0) - \frac{(1+\nu)^2}{4\pi Eh}\cdot\frac{x}{r}(f_4 u_1 + f_3 v_1) + + \frac{k}{2\pi Eh}\int_0^x \left[f_4(u_0+v_0) + f_3(v_0-u_0)\right]dx; \tag{16a}$$

$$v = -\frac{(1+\nu)^2}{4\pi Eh}\cdot\frac{y}{r}(f_4 u_1 + f_3 v_1) + \frac{k}{2\pi Eh}\int_0^x \left[f_4(u_0+v_0) + \right.$$

$$+ f_3(v_0 - u_0)]\,dx - \frac{1}{4\pi k^2R^2D}\int_0^x\int_0^y (f_1v_0 + f_2u_0)\,dx\,dy; \tag{16b}$$

$$w_x = -u_z; \tag{16c}$$

$$T_1 = \frac{1+\nu}{4\pi}\cdot\frac{x^2}{r^3}(f_4u_1 + f_3v_1) + \frac{1+\nu}{4\pi}\cdot\frac{k}{\sqrt{2}}\cdot\frac{y^2}{r^2}(f_4u_1' + f_3v_1') + + \frac{k}{4\pi}\left\{f_4(u_0+v_0) + f_3(v_0-u_0) + \frac{x}{r}\sqrt{2}(f_1u_0' - f_2v_0')\right\}; \tag{16d}$$

$$T_2 = -\frac{1+\nu}{4\pi}\left[\frac{x^2}{r^3}(f_4u_1 + f_3v_1) + \frac{k}{\sqrt{2}}\cdot\frac{y^2}{r^2}(f_4u_1' + f_3v_1')\right] - -\frac{\nu k}{4\pi}\left[f_4(u_0+v_0) + f_3(v_0-u_0) + \frac{x}{r}\sqrt{2}(f_1u_0' + f_2v_0')\right]; \tag{16e}$$

$$S = -\frac{1+\nu}{8\pi}\left\{k\frac{xy}{r^2}[f_4(u_2+v_5) + f_3(v_2-u_2)] - \frac{y}{r}\sqrt{2}(f_1u_0' - - f_2v_0')\right\}, \tag{16f}$$

$$M_1 = -\frac{1-\nu^2}{8\pi kR}\left[\frac{x^2}{kr^3}(f_3u_1 - f_4v_1) + \frac{y^2}{r^3}\cdot\frac{1}{\sqrt{2}}(f_3u_1' - f_4v_1')\right] - -\frac{\nu}{4\pi kR}\cdot\frac{x}{r}\sqrt{2}(f_1v_0' + f_2u_0'); \tag{16g}$$

$$M_2 = \frac{1-\nu^2}{8\pi kR}\left[\frac{x^2}{kr^3}(f_3u_1 - f_1v_1) + \frac{y^2}{r^3}\cdot\frac{1}{\sqrt{2}}(f_3u_1' - f_4v_1') + + f_3(u_0+v_0) + f_4(u_0-v_0)\right] - \frac{1+\nu^2}{8\pi kR}\cdot\frac{x}{r}\sqrt{2}(f_1v_0' + f_3u_0'); \tag{16h}$$

$$H = -\frac{1-\nu^2}{16\pi kR}\frac{xy}{r^2}[f_3(v_2+u_2) + f_4(u_2-v_2)] + \frac{y}{r}\sqrt{2}(f_1v_0' + + f_2u_0')\cdot\frac{(1-\nu)^2}{16\pi Eh}; \tag{16i}$$

$$Q_1 = \frac{1+\nu}{8\pi R}\left\{\frac{x^2}{r^3}\cdot\frac{\sqrt{2}}{k}(f_1v_0' + f_2u_0') - \frac{y^2}{r^3}\cdot\frac{1}{\sqrt{2}}[f_1(u_1'-v_1')] - - f_2(u_1'+v_1')]\right\} - \frac{\nu}{4\pi R}\left[f_1u_0 - f_2v_0 + \frac{x}{r}(f_3v_1 + f_4u_1)\right]; \tag{16j}$$

$$Q_2 = -\frac{1-\nu}{8\pi R}(f_4u_1 + f_3v_1) - \frac{1+\nu}{8\pi R}\cdot\frac{xy}{r^2}(f_1u_2 - f_2v_2); \tag{16k}$$

(c) From (7a) - (7f) /335

$$u_y = v_x;\quad w_y = -v_z; \tag{17a}$$

$$v = \frac{(3-\nu)(1+\nu)}{4\pi Eh}(f_1u_0 - f_2v_0) + \frac{(1+\nu)^2}{4\pi Eh}\cdot\frac{x}{r}(f_4u_1 + f_3v_1) + + \frac{1+2\nu}{2\pi Eh}k\int_0^x[f_3(u_0-v_0) - f_4(u_0+v_0)]\,dx - -\frac{k^2}{\pi Eh}\int_0^y\int_0^y(f_1v_0 + f_2u_0)\,dy\,dy + \frac{k^3}{\pi Eh}\int_0^x\int_0^y\int_0^y[f_3(u_0+v_0) - - f_4(v_0-u_0)]\,dx\,dy\,dy; \tag{17b}$$

$$T_1 = \frac{1-\nu}{8\pi}\cdot\frac{y}{r}k\sqrt{2}(f_1u_0' - f_2v_0') - \frac{1+\nu}{8\pi}k\frac{xy}{r^2}[f_4(u_2+v_2) + + f_3(v_2-u_2)] + \frac{k^2}{2\pi}\int_0^x\frac{y}{r}(f_3u_1 - f_4v_1)\,dx; \tag{17c}$$

$$T_2 = \frac{3+\nu}{8\pi}k\frac{y}{r}[f_1(u_1+v_1) + f_2(u_1-v_1)] - \frac{1+\nu}{4\pi}k\frac{xy}{r^2}(f_4u_2 + + f_3v_2); \tag{17d}$$

$$S=-\frac{1+\nu}{4\pi}\left[\frac{x^2}{r^2}(f_4u_1+f_3v_1)+\frac{y^2}{r^2}\frac{k}{\sqrt{2}}(f_4u_1'+f_3v_1')\right]+$$
$$+\frac{k}{4\pi}\left[\frac{x}{r}\sqrt{2}(f_1u_0'-f_2v_0')+f_4(u_0+v_0)+f_3(v_0-u_0)\right]; \tag{17e}$$
$$M_1=-\frac{3+\nu^2}{16\pi kR}\cdot\frac{y}{r}\sqrt{2}(f_1v_0'-f_2u_0')+\frac{1-\nu^2}{16\pi kR}\cdot\frac{xy}{r^2}[f_3(u_2+v_2)-$$
$$-f_4(v_2-u_2)]-\frac{\nu}{4\pi R}\int_0^x\frac{y}{r}(f_3v_1+f_4u_1)\,dx; \tag{17f}$$
$$M_2=\frac{1-4\nu-\nu^2}{16\pi kR}\cdot\frac{y}{r}\sqrt{2}(f_1v_0'+f_2u_0')-\frac{1-\nu^2}{16\pi kR}\frac{xy}{r^2}[f_3(u_2+v_2)-$$
$$-f_4(v_2-u_2)]-\frac{1}{4\pi kR}\int_0^x\frac{y}{r}(f_3v_1+f_4u_1)\,dx; \tag{17g}$$
$$H=\frac{1-\nu^2}{8\pi k^2R}\left[\frac{x^2}{r^2}(f_3u_1-f_4v_1)-\frac{y^2}{r^2}\cdot\frac{k}{\sqrt{2}}(f_3u_1'-f_4v_1')\right]+$$
$$+\frac{1-\nu}{8\pi kR}\left[\sqrt{2}\frac{x}{r}(f_1v_0'+f_2u_0')-f_4(u_0-v_0)+f_3(u_0+v_0)\right]; \tag{17h}$$
$$Q_1=\frac{3+\nu}{8\pi R}\cdot\frac{y}{r}(f_3v_1+f_4u_1)-\frac{1+\nu}{8\pi R}\cdot\frac{xy}{r^2}(f_1u_2-f_2v_2); \tag{17i}$$
$$Q_2=-\frac{1+\nu}{8\pi kR}\left\{-\frac{x^2}{r^2}\sqrt{2}(f_1v_0'+f_2u_0')+\frac{y^2}{r^2}\cdot\frac{1}{\sqrt{2}}[f_1(u_1'-v_1')-\right.$$
$$\left.-f_2(u_1'+v_1')]\right\}+\frac{1}{4\pi R}\left[f_1u_0-f_2v_0-\frac{x}{r}(f_4u_1+f_3v_1)\right]. \tag{17j}$$ /336

Here we have

$$f_1=\mathrm{ch}\frac{kx}{2}\cos\frac{kx}{2};\quad f_2=\mathrm{sh}\frac{kx}{2}\sin\frac{kx}{2}; \tag{18a}$$
$$f_3=\mathrm{sh}\frac{kx}{2}\cos\frac{kx}{2};\quad f_4=\mathrm{ch}\frac{kx}{2}\sin\frac{kx}{2}; \tag{18b}$$
$$u_n=\mathrm{ker}_n\left(\frac{kr}{\sqrt{2}}\right);\quad v_n=\mathrm{kei}_n\left(\frac{kr}{\sqrt{2}}\right); \tag{18c}$$
$$\mathrm{ker}_n(z)+i\,\mathrm{kei}_n(z)=i^{-n}K_n(z\sqrt{i});\quad z=\frac{kr}{\sqrt{2}}, \tag{18d}$$

where $K_n(z\sqrt{i})$ is a modified Bessel function of the second kind. When calculating the integrals, we employ the theory of generalized functions, as was done above (Ref. 3).

Rectangular plate. In this case $R_2 = \infty$, $k = 0$.

In (10a) - (12f) or in (15a) - (17j), let us pass to the limit in the case $k \to 0$ and $R_2 \to \infty$. Employing the values of the functions (18a) - (18d) for small values of the argument, we obtain the following after certain simplifications:

(a)

$$\left.\begin{aligned} &w=\frac{1}{8\pi D}r^2\ln r;\\ &M_1=-\frac{1+\nu}{4\pi}\ln r+\frac{1-\nu}{4\pi}\frac{x^2}{r^2};\quad M_2=-\frac{1+\nu}{4\pi}\ln r-\frac{1-\nu}{4\pi}\frac{x}{r^2};\\ &H=-\frac{1-\nu}{4\pi}\cdot\frac{xy}{r^2},\quad Q_1=-\frac{1}{2\pi}\cdot\frac{x}{r^2},\quad Q_2=-\frac{1}{2\pi}\cdot\frac{y}{r^2};\end{aligned}\right\} \tag{19}$$

(b)

$$u = -\frac{(3-\nu)(1+\nu)}{4\pi Eh}\ln r + \frac{(1+\nu)^2}{4\pi Eh}\cdot\frac{x^2}{r^2}; \tag{20a}$$

$$v = \frac{(1+\nu)^2}{4\pi Eh}\cdot\frac{xy}{r^2}; \tag{20b}$$

$$T_1 = \frac{1}{4\pi}\frac{x}{r^2}\left[2(1+\nu)\frac{y^2}{r^2} - 3 - \nu\right], \quad T_2 = -\frac{1}{4\pi}\frac{x}{r^2}\left[2(1+\nu)\frac{y^2}{r^2} - 1 + \nu\right]; \tag{20c}$$

$$S = -\frac{1}{4\pi}\frac{y}{r^2}\left[2(1+\nu)\frac{x^2}{r^2} + 1 - \nu\right]; \tag{20d}$$

(c)

$$u = \frac{(1+\nu)^2}{4\pi Eh}\cdot\frac{xy}{r^2}; \tag{21a}$$

$$v = -\frac{(3-\nu)(1+\nu)}{4\pi Eh}\ln r - \frac{(1+\nu)^2}{4\pi Eh}\cdot\frac{x^2}{r^2}; \tag{21b}$$

$$T_1 = -\frac{1}{4\pi}\cdot\frac{y}{r^2}\left[2(1+\nu)\frac{x^2}{r^2} - 1 + \nu\right]; \tag{21c}$$

$$T_2 = \frac{1}{4\pi}\cdot\frac{y}{r^2}\left[2(1+\nu)\frac{x^2}{r^2} - 3 - \nu\right]; \tag{21d}$$

$$S = -\frac{1}{4\pi}\cdot\frac{x}{r^2}\left[2(1+\nu)\frac{y^2}{r^2} + 1 - \nu\right]. \tag{21e}$$

The expressions obtained for stresses and displacements coincide with the well known solution of Love. /337

In conclusion, we would like to note that from (10a) - (10f) and (15a) - (17j) we may readily obtain the asymptotic values of displacements and internal stresses close to the point at which the concentrated force is applied (Ref. 9) which have a simpler form.

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INVESTIGATION OF THE BRITTLE FRACTURE OF SAMPLES WITH STRESS CONCENTRATORS /338

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It is well known that the strength of samples made of brittle materials having cuts is considerably less than the strength of smooth samples having the same cross-section, due to a sharp local increase in stresses in the vicinity of the cuts. For ideally brittle and homogeneous materials, this decrease in strength is taken into account by the theoretical coefficient of the stress concentration k_T (Ref. 2, 4). However, there is significant discrepancy between the theoretical and the so-called effective coefficient of stress concentration $k_{\text{Э}}$ determined experimentally. This discrepancy is primarily related to the non-uniformity and deffective nature of the structure, which is inherent to all materials to a certain extent. Structural imperfections may produce significant, and a rapidly damped local disturbance of the stress state. In a completely similar manner, high local and rapidly damped stresses arise for the stress concentrators which we derived. Therefore, when the structure is correspondingly non-uniform, artificial concentrators having specific dimensions can have only a slight influence upon the strength of an element, or may have absolutely no influence upon it. In this report we shall deal with the decreased sensitivity of a material to the stress concentration characterized by the coefficient $q = \dfrac{k_{\text{Э}} - 1}{k_T - 1}$. Thus, in actual designs for strength, the nonuniformity of the structure must be taken into account, along with allowance for the stress concentrators applied.

Figure 1

For this purpose, this article advances a macroscopic hypothesis of brittle fracture (Ref. 1) corresponding to a structural imperfection in integral terms. The influence of these imperfections may be neutralized within the limits of a certain specific volume contained in /339 a sphere having the radius ρ. We may intrepret the sphere having the radius ρ as the minimum volume of the given material which -- on the basis of the laws of statics -- has mechanical properties which may be determined by customary tests. The value of the parameter ρ depends on the magnitude of the structural nonuniformities of the material, their nature, and the distribution density. The larger is ρ, the more coarse-grained and nonuniform is the structure, and the greater are the microdefects in the given material. The parameter ρ provides the basis for determining the macrodeformations of a real body, which are related to the macrostresses by Hooke's law. Thus, the hypothesis of brittle fracture may be formulated as follows. Brittle fracture occurs in a given

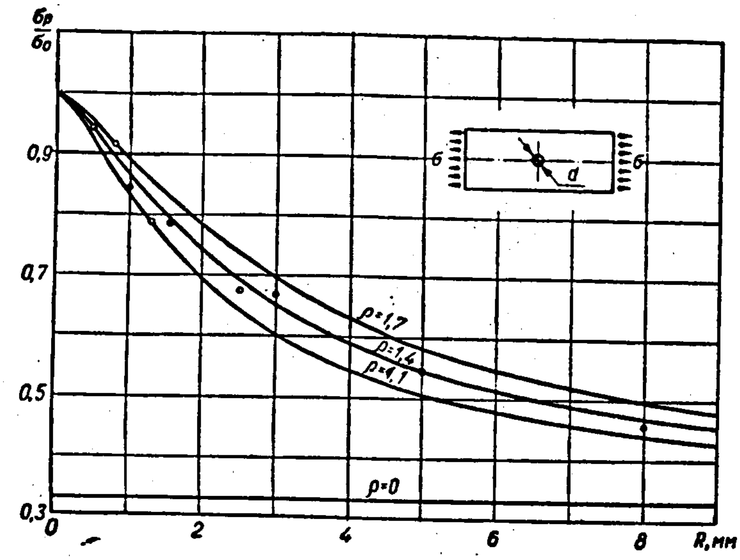

Figure 2

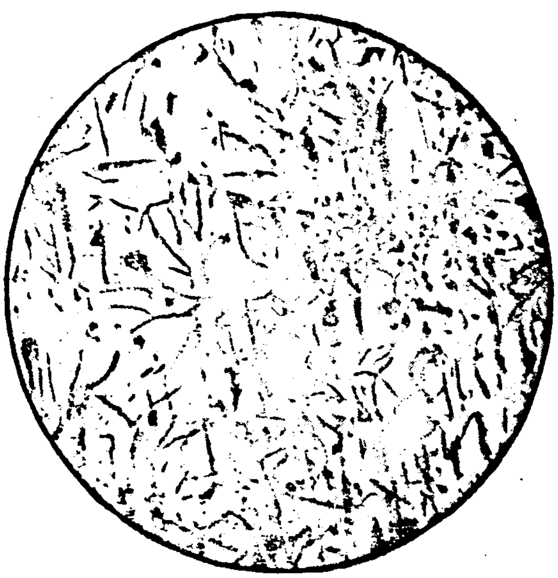

Figure 3

volume when the macrostresses reach the separation resistance limit of the given material, which may be determined when smooth samples are subjected to tension. Consequently, the concentration coefficient of macrostresses must coincide with the effective stress concentration coefficient obtained experimentally, which will provide the basis upon which the research results below are analyzed.

This article investigates the influence of the stress concentration on the supporting capacity of samples made of brittle microscopically nonuniform materials.

The material used for the study was gray pig iron which had sufficient brittleness and a very nonuniform structure produced by graphite inclusions which represented natural stress concentrators. After normilization or hardening in oil, the gray pig iron had minimum plastic deformation (no more than 0.2%). Tests were performed for torsion (Ref. 3), which showed that the samples chosen for the study were quite brittle.

Strength of plates with circular holes under tension. The test was performed on plates made of gray pig iron, with the brand name SCh 12-28 (the microsection x 200 is shown in Figure 1) with circular holes having a different diameter (0.7; 1.0; 1.4; 2.5; 3.0; 5.0; 6.0; 10.0 and 16.0 mm). The plate dimensions (120 x 400 x 2 mm) were selected so that the influence of its edges upon the stress state at the hole was negligibly small. The plate was ruptured in special clamps which prevented their fracture when being mounted and which provided a uniform loading distribution. The experimental results shown in Figure 2 (the small circles are for the first group of samples, and the dots are for the second group of samples) clearly indicate the dependence of the breaking load on the hole dimensions.

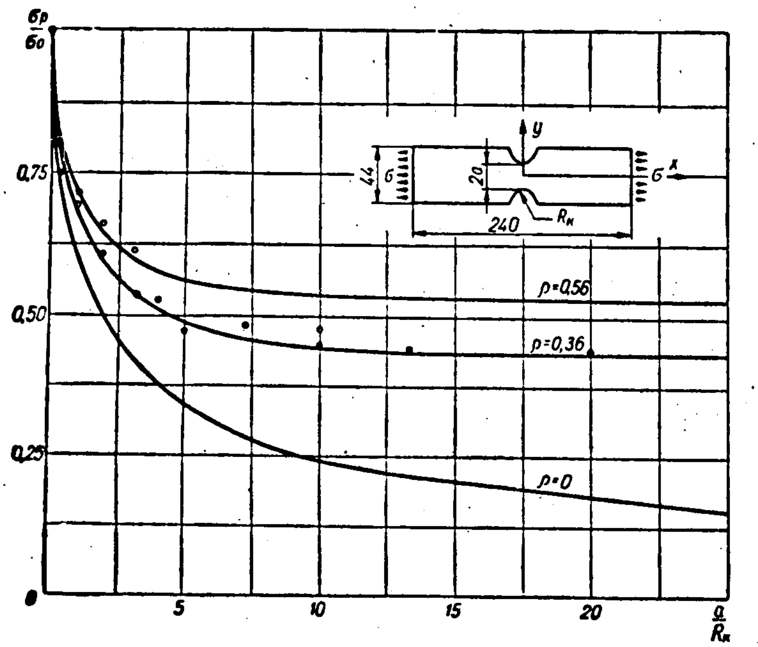

Figure 4

The breaking load was also determined on the basis of the macrostress hypothesis according to the following formula (Ref. 1)

$$\sigma_p = \frac{\sigma_0}{k}, \tag{1}$$

where σ_0 is the resistance of the material to separation; k -- macrostress concentration coefficient /341

$$k = \frac{2\nu a^2}{(1+\nu)(1+a)^2(1+2a+2a^2)} + \\ + \frac{3+11a+25a^2+40a^3+42a^4+24a^5+8a^6}{(1+2a+2a^2)^3}; \tag{2}$$

$$a = \frac{\rho}{R}.$$

As may be seen from formulas (2), the macroscopic concentration coefficient depends both on the structural nonuniformity of the material (ρ), and on the size of the hole (R). A comparison of the experimental data with the theoretical data enables us to determine the structural strength parameter for the material being studied, which is ρ = 1.4 mm ($\sigma_0 \approx 17$ dyne/mm^2).

The results of the experimental research provide good substantiation of the general pattern of the plate strength change as a function of the concentrator radius, which was predicted by the macroscopic hypothesis of brittle fracture.

As analytical calculations as well as experimental data have shown, holes whose diameter is significantly less than the parameter ρ have hardly any influence upon the plate strength. When the hole diameter increases, the effective coefficient of stress concentration $k_{э}$ rapidly increases, and then slowly approaches the theoretical value $k_T = 3$.

Strength of bands with hyperbolic cuts under tension. The influence of stress concentration upon strength, with allowance for structural nonuniformity, was also investigated for bands with hyperbolic cuts under uniaxial tension.

Two groups of bands with the dimensions 44 x 240 x 2 mm were prepared, respectively, from two different melts of gray pig iron, with the brand name SCh 24-48 (the microsection x 200 is shown in Figure 3). Cuts of a hyperbolic profile (Figure 4) which may be described by the equation

$$y = \sqrt{\frac{(x^2 - 64) R_\kappa}{a}},$$

with different radii of curvature at the apex (R_K = 0.5; 0.6; 0.8; 1.1; 1.6; 2.0; 2.5; 4.0; 8.0 and 16.0 mm) for a constant width of the minimum transverse cross-section 2a = 16.00 mm, were drawn on a copying machine within an accuracy of $\pm$ 0.05 mm. The bands were subjected to fracture in order to determine the breaking load. This load was also determined analytically. For this purpose, the displacements were found from the stress functions of Neyber (Ref. 2), and then the macrostresses were determined in bands with hyperbolic cuts under tension. We may obtain the breaking load as a function of the resistance to separation σ_0, the geometry of the concentrator, characterized by the parameter /343
κ, and the structural parameter ρ from (1). We may determine k as:

$$k = \frac{2\left[\bar{u} + \frac{1+\nu}{2}\left(\sin^2\bar{v} - \frac{1}{1+\kappa^2}\right)\frac{\operatorname{sh}\bar{u}\operatorname{ch}\bar{u}}{\operatorname{sh}^2\bar{u} + \cos^2\bar{v}} + \frac{\nu\eta}{\sqrt{\kappa^2 + \eta(2-\eta)}}\right]}{\eta(1+\nu)\left(\operatorname{arcctg}\kappa + \frac{\kappa}{\kappa^2+1}\right)}, \tag{3}$$

where

$$\operatorname{sh}^2\bar{u} = \frac{1}{1+\kappa^2}\left[\sqrt{\eta^2(1-\eta)^2 + \left(\frac{\kappa^2}{2}+\eta\right)^2} - \eta(1-\eta) - \frac{\kappa^2}{2}\right]; \tag{4a}$$

$$\operatorname{ch}^2\bar{u} = \frac{1}{1+\kappa^2}\left[\sqrt{\eta^2(1-\eta)^2 + \left(\frac{\kappa^2}{2}+\eta\right)^2} - \eta(1-\eta) + \frac{\kappa^2}{2} + 1\right]; \tag{4b}$$

$$\cos^2\bar{v} = \frac{1}{1+\kappa^2}\left[\sqrt{\eta^2(1-\eta)^2 + \left(\frac{\kappa^2}{2}+\eta\right)^2} + \eta(1-\eta) + \frac{\kappa^2}{2}\right]; \tag{4c}$$

$$\sin^2\bar{v} = \frac{1}{1+\kappa^2}\left[-\sqrt{\eta^2(1-\eta)^2 + \left(\frac{\kappa^2}{2}+\eta\right)^2} + \eta(1-\eta) + \frac{\kappa^2}{2} + 1\right]; \tag{4d}$$

$$\kappa = \sqrt{\frac{R_\kappa}{a}}, \quad \eta = \frac{\rho}{a}; \tag{4e}$$

where R_K is the radius of curvature at the concentrator apex.

Figure 4 shows the change in the breaking load referred to the separation resistance $\frac{p}{\sigma_0}$, as a function of $\frac{a}{R_K}$ for the values $\rho = 0.36$ and $\rho = 0.56$ mm.

Figure 4 shows the values obtained experimentally: the small circles -- for the first group, and the dots for the second group of samples. These represent the mean values for no less than three tests.

The structural parameter of the strength $\rho = 0.37$ mm ($\sigma_0 = 27$ dyne/mm^2) may be determined by comparing the experimental data with theoretical data. We should stress the good agreement between the experimental data and the theoretical curve.

The research reveals the following.

1. The nonuniformity of the structure is the most important factor which determines the strength of elements made of brittle materials, containing artificial stress concentrators. This factor may be considered analytically on the basis of the macroscopic hypothesis of brittle fracture, which has been experimentally corroborated.

2. A comparison of the mechanical and metallographic research shows that the coarser is the structure -- particularly, graphite inclusions -- the larger is the structural strength parameter, and, consequently, the smaller is the influence of artificial stress concentrators on the element strength. The /344
structural nonuniformity also specifies the strength limit of the material: for $\rho = 1.4$ mm, $\sigma_0 = 17$ dyne/mm^2, and for $\rho = 0.37$ mm, $\sigma_0 = 27$ dyne/mm^2.

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